The Split Delivery Vehicle Routing Problem with Time Windows and Customer Inconvenience Constraints

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Abstract

In classical routing problems, each customer is visited exactly once. By contrast, when allowing split deliveries, customers may be served through multiple visits. This potentially results in substantial savings in travel costs. Even if split deliveries are beneficial to the transport company, several visits may be undesirable on the customer side: at each visit the customer has to interrupt his primary activities and handle the goods receipt. The contribution of the present paper consists in a thorough analysis of the possibilities and limitations of split delivery distribution strategies. To this end, we investigate two different types of measures for limiting customer inconvenience (a maximum number of visits and the temporal synchronization of deliveries) and evaluate the impact of these measures on carrier efficiency by means of different objective functions (comprising variable routing costs, costs related to route durations, fixed fleet costs). We consider the vehicle routing problem with time windows in which split deliveries are allowed (SDVRPTW) and define the corresponding generalization that takes into account customer inconvenience constraints (SDVRPTW-IC). We design an extended branch-and-cut algorithm to solve the SDVRPTW-IC and report on experimental results showing the impact of customer inconvenience constraints. We finally draw useful insights for logistics managers on the basis of the experimental analysis carried out.

Key words: Split delivery vehicle routing problem, Time windows, Synchronization, Maximum number of visits, Branch-and-cut

1. Introduction

In classical routing problems concerning the delivery of goods, each customer is visited exactly once. By contrast, when allowing split deliveries, customers may be served by means of multiple visits. This potentially results in substantial savings in travel costs and fleet size, as in the split delivery vehicle routing problem (SDVRP), the relaxation of the vehicle routing problem (VRP) in which split deliveries are possible (see Archetti and Speranza (2012) and Irnich *et al.* (2014) for recent surveys on the topic). The option of split deliveries is clearly beneficial to the transport company. On the customer side, though, several visits cause inconvenience, as at each visit, the customer has to interrupt his primary activities to handle the goods receipt.

In the paper at hand, we introduce generalizations of the SDVRP that allow to control the degree of inconvenience caused by split deliveries and to balance overall distribution costs and customer satisfaction. This creates a win-win situation for transport companies and their customers. We examine two measures for limiting customer inconvenience:

(i) Maximum number of visits: this is the obvious and most direct way to limit customer inconvenience.
(ii) Temporal synchronization of deliveries: it is required that all deliveries to the same customer arrive within a pre-defined time span.

Maximum Number of Visits. When a customer's demand exceeds the vehicle capacity, this customer is certainly split, so that the minimum number of visits to any customer is $n_i^{\min} = \lceil d_i/Q \rceil$ (where d_i is the

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demand of customer i and Q the vehicle capacity). Archetti *et al.* (2006b) compare different VRP variants that result from fixing the number of visits to this minimum. Let VRP$^+$ be the variant in which each customer i is visited exactly $n_i^{\min}$ times, where for $n_i^{\min} > 1$ the demand d_i can be arbitrarily split among the $n_i^{\min}$ visits. The authors show that, compared to the optimal VRP$^+$ solution, cost savings of 50% are possible when allowing an arbitrary number of visits, and that this bound is tight. By allowing more than the minimum number of visits, a large number of intermediate SDVRP variants can be defined, all with the purpose of controlling the possible customer inconvenience: for each customer i, the number of visits to this customer can be bounded above by $n_i^{\max} \geq n_i^{\min}$. Moreover, one may limit the overall number of visits to $n^{\max}$ for any $n^{\max} \geq \sum_i n_i^{\min}$ in order to reduce customer inconvenience.

Salazar-González and Santos-Hernández (2015) introduce the *split-demand one-commodity pickup-and-delivery traveling salesman problem (SD1PDTSP)*, a very general problem that, despite its name, encompasses the multi-vehicle SDVRP as well as several other capacitated and uncapacitated routing problems *without time windows* as special cases. The authors propose a compact formulation for the SD1PDTSP and model the requirement of a maximum number of visits in the underlying network, by creating $n_i^{\max}$ vertices for each customer i.

Temporal Synchronization of Deliveries. In this paper, we introduce synchronized deliveries as an alternative measure to reduce customer inconvenience. For this purpose, we embed synchronization constraints into a new split delivery routing problem which guarantees that all split deliveries occurring to a customer must take place in a time interval of a given maximum duration. As the time dimension is relevant then, we focus on the *split delivery vehicle routing problem with time windows* (SDVRPTW), which is the split-delivery relaxation of the vehicle routing problem with time windows (VRPTW, Desaulniers *et al.*, 2014). The variant of the SDVRPTW in which synchronization constraints are embedded is denoted by SDVRPTW-S; it is a special case of the more general SDVRPTW-IC that we formally define in Section 3.

Minimum Delivery Amounts. When trying to minimize customer inconvenience, what counts from the customer's point of view is the number of interruptions of his primary activities, in other words, the number of visits. A third way to reduce the number of interruptions is to require that split deliveries are allowed only if a minimum fraction of the customer's demand is delivered at each visit. Gulczynski *et al.* (2010) consider a pertinent generalization of the SDVRP. Besides defining a heuristic method for solving the problem, the authors give bounds for a worst-case SDVRP-MDA scenario. Their results are extended in Xiong *et al.* (2013). In the context of routing problems with profits, the idea of allowing to serve a customer by means of multiple visits only if a minimum fraction of the customer's demand is served at each visit is further examined by Wang *et al.* (2014). We do not consider the option of specifying minimum delivery amounts in our study, for two reasons. First, minimum delivery amounts are only an indirect way to achieve the primary goal of limiting the number of visits. It is simpler and more intuitive to set such a number directly. Second, and even more importantly, a minimum delivery amount does not make sense when the service times at customers can be assumed to be independent of the amount delivered. Judging from our experience, this is the case in many (though not all) real-world situations; moreover, it is a common assumption in the literature on the SDVRPTW as reviewed in the next paragraph.

To our knowledge, the most effective exact algorithms for the solution of the SDVRPTW are the branch-and-price-and-cut algorithms proposed by Archetti *et al.* (2011b) and Luo *et al.* (2016) (which are based on the work of Desaulniers, 2010), and the branch-and-cut algorithm proposed by Bianchessi and Irnich (2016). The cited solution approaches are able to solve slightly different subsets of the SDVRPTW benchmark instances. However, concerning the number of instances solved to optimality, the branch-and-cut algorithm proposed in (Bianchessi and Irnich, 2016) is superior, solving 5% more instances than the other solution approaches. In this work, we extend this branch-and-cut algorithm to address the different special cases of the SDVRPTW-IC.

The contribution of this paper is not only innovative from a methodological point of view. Even more importantly, we shed light on complex interdependencies between VRPTW, SDVRPTW, and SDVRPTW-IC special cases. Indeed, straightforward comparisons carry the danger of not taking all relevant effects into account. The standard SDVRPTW objective is the minimization of the variable routing costs (Desaulniers, 2010). The most important insight gained from our experiments with the SDVRPTW-IC is that an exclusive comparison on the basis of variable routing costs is insufficient. Overall logistics costs surely depend on

(i) variable routing costs,
(ii) costs related to route durations, and

(iii) costs of the employed fleet,

and these cost elements should be included in a meaningful study analyzing savings that result from split deliveries.

To underline this statement, we present, at this early stage, the following brief computational comparison of VRPTW and SDVRPTW solutions. We used the well-known benchmark set of Solomon (1987), both as VRPTW and SDVRPTW instances. In order to keep the computational effort manageable, we considered only the smaller-sized instances constructed with the subsets of the first 25 and 50 customers respectively. However, as always done for the SDVRPTW, the vehicle capacity Q is varied ($Q = 30, 50$ and 100) leading to $3 \cdot 2 \cdot 56 = 336$ instances (more details are provided in Section 5). With the standard objective of minimizing the variable routing costs and the branch-and-cut that will be presented in Section 4, we obtained the results summarized in Table 1. The columns *Feas.* and *Opt.* show the number of instances for which a

Table 1: VRPTW and SDVRPTW solutions and comparison

Instances		**VRPTW**		**SDVRPTW**		**Comparison**				
n	#	Feas.	Opt.	Feas.	Opt.	#	Rout. Costs (↓ / =)	Durations (↓ / = / ↑)	#Vehicles (↓ / =)	Dominating (Pareto)
25	168	135	135	168	168	135	56/79	10/79/46	8/127	10/135
50	168	112	66	168	95	64	39/25	8/25/31	1/63	8/64
Total	336	247	201	336	263	199	95/104	18/104/77	9/190	18/199

feasible VRPTW solution exists (recall that the capacity Q is lowered compared to Solomon's definition) and for which both an optimal VRPTW and an optimal SDVRPTW solution were computed. Only the instances solved to optimality as VRPTW and as SDVRPTW were considered in the comparison. For these 199 instances, the section *Comparison* shows the number of instances in which the SDVRPTW solution improved (↓) the corresponding VRPTW solution w.r.t. variable routing costs (*Rout. Costs*), route durations (*Durations*), called "schedule times" in the work of Solomon (1987), and the number of vehicles employed (*#Vehicles*). Recall that the routing costs of the SDVRPTW solution cannot increase but may stay constant (=). In our experiments, the SDVRPTW solution did never employ more vehicles than the corresponding VRPTW solution (this is why there are only the two cases ↓ and = in column *#Vehicles*). Dominating SDVRPTW solutions (their number is reported as *Dominating*) are those for which one of the three criteria is strictly improved while the others are not worse.

Beyond the numbers reported in Table 1, there are some important findings:

(i) For only 7 of the 199 instances, the variable routing costs are reduced by more than 1.5%.

(ii) For the 9 instances for which the number of vehicles decreased, it decreased by 1.

(iii) For 171 instances, the variable routing costs were reduced by less than 0.5%.

Additionally, Figure 1 quantifies, for the 95 instances for which variable routing costs decreased, the relationship between savings in variable routing costs and deviations of the route durations. To integrate the third criterion, we distinguish between SDVRPTW solutions that save (at least) one vehicle and all other solutions. The figures seem to indicate that, in many cases, even a rather small reduction in variable routing costs leads to a notable increase of the route durations. Recall, however, that such a statement is based on a limited set of benchmark problems and, more seriously, route durations and required fleet size are just an outcome of a pure variable routing costs minimization. We draw the following conclusions from the presented comparison of VRPTW and SDVRPTW:

(i) As the scientific VRP literature has not yet studied the full interdependency between all relevant cost types, a new SDVRPTW model should consider cost components related to route durations, such as driver wages, and fleet-related costs in addition to variable routing costs. This provides a more complete picture of the overall logistics costs and allows managers to better foresee the consequences of a possible change of the delivery strategy.

(ii) The incorporation of constraints that reduce customer inconvenience creates a variety of VRP models, for which VRPTW and SDVRPTW are the extreme cases. It is necessary to study these variants with the aim to better understand the impact of the different inconvenience constraints on the relevant cost types.

(iii) For the Solomon-based SDVRPTW benchmark set, we have seen that the decrease in routing costs is only marginal compared to an offered 50% savings discussed in worst-case analyses. It is known that the savings from split deliveries mainly depend on the demand distribution (Archetti *et al.*, 2006b).

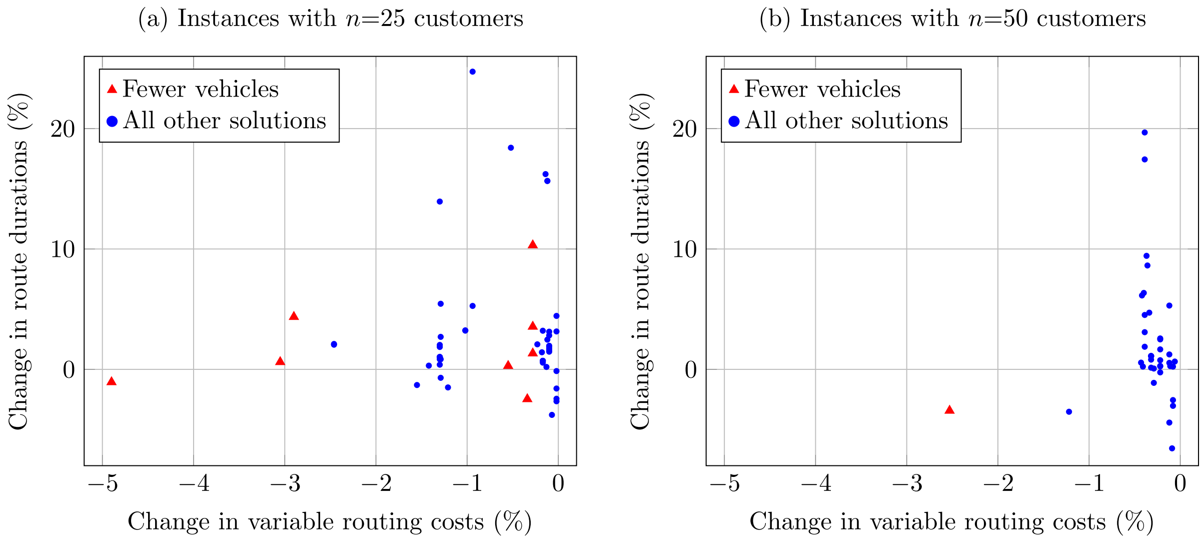


Figure 1: SDVRPTW vs. VRPTW solutions: relationship between savings in variable routing costs, change of route durations, and reduction of the number of routes.

Without specific patterns for the customers' demand realizations, the Solomon-based benchmarks lack generality. We therefore create a new benchmark set in which groups of instances are characterized by different demand distributions (see Section 5.1).

The remainder of the paper is organized as follows. In Section 2, we formally define the SDVRPTW and list some important properties of the problem. A mathematical model for the SDVRPTW-IC is then discussed in Section 3. In Section 4, we present the branch-and-cut algorithm designed to solve the SDVRPTW-IC. Based on the experimental results obtained, we present in Section 5 the analysis of the impact of inconvenience constraints. Final conclusions are drawn in Section 6.

2. The SDVRPTW and Properties of Optimal Solutions

Let us first recall the definition of the SDVRPTW. The problem can be defined on a directed graph $G = (V, A)$, with vertex set V and arc set A. The vertex set V contains vertices 0 and $n+1$, representing the depot at the beginning and the end of the planning horizon respectively, and the set $N = \{1, \dots, n\}$ representing the n customers. Each customer $i \in N$ is associated with a positive demand d_i that must be delivered by means of one or more visits within a prescribed time window $[e_i, l_i]$. Each delivery at customer i must start within $[e_i, l_i]$, but a vehicle may arrive prior to e_i and then wait until e_i before starting the delivery. Moreover, a time window $[e_0, l_0] = [e_{n+1}, l_{n+1}]$ is associated with the depot to model the planning horizon. Each arc $(i, j) \in A$ represents the possibility to move from the location corresponding to vertex i to the location corresponding to vertex j, and it is associated with a non-negative travel time t_{ij} and a non-negative routing cost c_{ij}. In particular, t_{ij} includes the service time at i. We assume that the service time is constant for each visit and independent of the amount delivered. For each pair of vertices $i, j \in V, i \neq j$, there exists an arc $(i, j) \in A$ if $e_i + t_{ij} \leq l_j$. We assume that all customer time windows are reduced so that $e_i \geq e_0 + t_{0i}$ and $l_i \leq l_{n+1} - t_{i,n+1}$ holds for all $i \in N$. As is common, the set A includes the arc $(0, n+1)$, associated with zero travel time and routing cost, that allows modeling an idle vehicle, but not the arc $(n+1, 0)$. A fleet K of $|K|$ identical vehicles with a capacity Q is available to serve the customers. The vehicles are initially located at the depot. A route corresponds to a path from 0 to $n+1$ in G. A route is feasible if the total demand delivered at the visited customers does not exceed the vehicle capacity and the time windows are respected. The SDVRPTW consists of determining a set of least-cost feasible routes such that all customer demands are met.

Given the above definitions and assumptions, and further assuming that the triangle inequality holds for routing costs and travel times, it is possible to prove that, for any SDVRP(TW) instance that has an optimal solution, there exists an optimal solution with the following properties:

Property 1. Two routes share at most one split customer (Dror and Trudeau, 1990).

Property 2. Each arc between two vertices representing customers is traversed at most once (Gendreau *et al.*, 2006).

Property 3. For each pair of reverse arcs between two customers at most one of them is traversed (Desaulniers, 2010).

Property 4. All routes are elementary (Desaulniers, 2010).

If, in addition, the vehicle capacity Q and all demands d_i for $i \in N$ are integer, then there exists an optimal solution to the SDVRPTW fulfilling Properties 1–4 and

Property 5. All delivery quantities are positive integers (Archetti *et al.*, 2006a, 2011a).

These properties are exploited in the branch-and-cut algorithm that we present in Section 4.

3. The SDVRPTW with Customer Inconvenience Constraints

The SDVRPTW-IC is the generalization of the SDVRPTW taking into account upper bounds on the number of visits, and synchronization constraints for split deliveries occurring to the same customer. More formally, the following parameters become part of the problem definition:

Maximum number of visits: $n_i^{\max}$ and $n^{\max}$ limit the number of visits to $i \in N$ and the overall number of visits respectively;

Temporal synchronization of deliveries: Δ_i limits the length of the time interval in which all deliveries to $i \in N$ must take place.

Moreover, the impact of these customer inconvenience constraints on the following types of distribution costs is taken into account in the SDVRPTW-IC objective function:

Variable routing costs: These are given for each arc $(i, j) \in A$ and are denoted by c_{ij}. They may also include a penalty p_i when a customer $i \in N$ is visited. In this case, $\sum_{i \in N} n_i^{\min} p_i$ is the unavoidable penalty.

Costs related to route durations: We denote by γ the time-to-cost ratio that, multiplied by the duration of a route, yields the duration-related costs.

Fixed vehicle costs: The fixed costs for using a vehicle are denoted by C.

We now describe two important characteristics of SDVRPTW-IC solutions.

Proposition 1. *Given an SDVRPTW-IC instance fulfilling the assumptions made in Section 2. If this instance has an optimal solution, and if both routing costs and travel times satisfy the triangle inequality, the following two properties hold:*
(a) There exists an optimal solution fulfilling Properties 1–4.
(b) If the vehicle capacity Q and all demands d_i for $i \in N$ are integer, then there exists an optimal solution fulfilling Properties 1–5.

Proof.
(a) The proof of Property 1 is analogous to the one given by Gendreau *et al.* (2006) for the SDVRPTW, which, in turn, is based on the one by Dror and Trudeau (1990) for the SDVRP. Properties 2 and 3 follow immediately from Property 1. Given the above assumptions, Property 4 is fulfilled because a feasible SDVRPTW-IC solution with a non-elementary route that visits a customer more than once remains feasible with non-increased costs if all but the last visit to this customer are removed.
(b) The proof of this property is analogous to the one given by Archetti *et al.* (2006a) for the SDVRP.

□

We remark that, as Gulczynski *et al.* (2010) have shown, these properties are no longer fulfilled when minimum delivery amounts are specified.

It is anything but straightforward to develop a practicable and computationally attractive compact formulation for the SDVRPTW-IC. Bianchessi and Irnich (2016) have analyzed the difficulties of devising one for the SDVRPTW. Their arguments apply just as well to the SDVRPTW-IC and shall thus be briefly discussed in the following. First, as customers can be visited by several vehicles, it is impossible to attach unique resource variables to the vertices, e.g., variables indicating the accumulated customer demand and the service time. Consequently, formulations using Miller-Tucker-Zemlin types of constraints for the update of resource variables (see Miller *et al.*, 1960) are not directly applicable in the split-delivery context. Second, using a three-index formulation, i.e., variables with vehicle indices, is not practicable either, as the resulting symmetries make any known branching scheme ineffective. Symmetry-breaking constraints (see, e.g. Fischetti *et al.*, 1995) can only mitigate the negative effects of symmetry. Third, the formulation proposed by van Eijl (1995) for the delivery man problem and the one by Maffioli and Sciomachen (1997) for the sequential ordering problem show that resource variables may be associated with arcs. However, even if we can exploit Property 2 and associate time variables with arcs between customers, the problem remains that arcs between depot and customers (or vice versa) may be traversed by more than one vehicle. Hence, no time variables that uniquely define the vehicle travel times can be associated with these arcs.

Notwithstanding the above objections, we subsequently present a three-index model for the SDVRPTW-IC fulfilling Properties 2–4. Because of the mentioned weaknesses of such a formulation, however, we do not try to solve this model directly. Its purpose is solely to give a complete formal description of the SDVRPTW-IC. Our solution approach to the SDVRPTW-IC is based on a relaxed compact formulation using two-index variables and is described in the next section. In both models, we do not require Property 1, because this property cannot well be formulated with linear constraints. Moreover, Property 5 is fulfilled whenever a basic solution to an instance with integer demands and vehicle capacity is given.

The following model can be seen as a multi-commodity network flow formulation with additional variables and constraints, with a commodity for each available vehicle. The formulation uses

(i) binary flow variables x^k_{ij} equal to 1 if vehicle $k \in K$ travels along arc $(i,j) \in A$, and 0 otherwise;

(ii) non-negative continuous flow variables T^k_i representing the start of service of vehicle $k \in K$ when visiting vertex $i \in N$;

(iii) non-negative continuous variables δ^k_i representing the quantity delivered by vehicle $k \in K$ to customer $i \in N$;

(iv) continuous variables E_i representing the earliest start of service at customer $i \in N$.

The symbols $\Gamma^+(S)$ and $\Gamma^-(S)$ respectively denote the forward and backward star of $S \subseteq N$. For simplicity, we use $\Gamma^+(i)$ and $\Gamma^-(i)$ whenever $S = \{i\}$. Moreover, we define $A(N) = \{(i,j) \in A : i \in N, j \in N\}$.

The multi-commodity flow formulation for the SDVRPTW-IC is as follows:

$$\min \quad \sum_{k\in K}\left(\sum_{(i,j)\in A} c_{ij}x^k_{ij} + \gamma\left(T^k_{n+1} - T^k_0\right) + C\sum_{i\in N} x^k_{0i}\right) \tag{1a}$$

$$\text{s.t.} \quad \sum_{(0,j)\in\Gamma^+(0)} x^k_{0j} = \sum_{(i,n+1)\in\Gamma^-(n+1)} x^k_{i,n+1} = 1 \qquad k \in K \tag{1b}$$

$$\sum_{(h,i)\in\Gamma^-(i)} x^k_{hi} - \sum_{(i,j)\in\Gamma^+(i)} x^k_{ij} = 0 \qquad i \in N,\ k \in K \tag{1c}$$

$$x^k_{ij}(T^k_i + t_{ij} - T^k_j) \le 0 \qquad (i,j) \in A,\ k \in K \tag{1d}$$

$$e_i \le T^k_i \le l_i \qquad i \in N, k \in K \tag{1e}$$

$$\sum_{k\in K} \delta^k_i \ge d_i \qquad i \in N \tag{1f}$$

$$0 \le \delta^k_i \le \min\{d_i, Q\} \sum_{(i,j)\in\Gamma^+(i)} x^k_{ij} \qquad i \in N,\ k \in K \tag{1g}$$

$$\sum_{i\in N} \delta^k_i \le Q \qquad k \in K \tag{1h}$$

$$x^k_{ij} \in \{0,1\} \qquad (i,j) \in A,\ k \in K \tag{1i}$$

Additional constraints enforcing Properties 2 and 3 are added:

$$\sum_{k\in K} x^k_{ij} \le 1 \qquad (i,j) \in A(N) \tag{1j}$$

$$\sum_{k\in K} x^k_{ij} + x^k_{ji} \le 1 \qquad (i,j),(j,i) \in A(N) : i < j \tag{1k}$$

Constraints to alleviate customer inconvenience are:

$$\sum_{k\in K}\sum_{(i,j)\in\Gamma^+(i)} x_{ij}^k \leq n_i^{\max} \qquad i \in N \tag{1l}$$

$$\sum_{k\in K}\sum_{i\in N}\sum_{(i,j)\in\Gamma^+(i)} x_{ij}^k \leq n^{\max} \tag{1m}$$

$$E_i \leq T_i^k \leq E_i + \Delta_i \qquad i \in N, k \in K \tag{1n}$$

The objective function (1a) calls for the minimization of the total variable routing costs, the costs related to route durations, and the fixed costs for employing vehicles. Constraints (1b) and (1c) impose the route associated with each vehicle to be a 0-$(n+1)$-path. Feasibility regarding time-window constraints and elementarity of the routes is guaranteed by (1d) and (1e). Clearly, constraints (1d) can be linearized by $T_i^k + t_{ij} - T_j^k \leq M_{ij}(1 - x_{ij}^k)$, where M_{ij} is an arc-specific large constant, e.g., $M_{ij} = \max\{l_i + t_{ij} - e_j, 0\}$. Constraints (1f) ensure customer demands are met. Constraints (1g) allow a vehicle to deliver only to visited customers and (1h) are the capacity constraints. The domain of the vehicle flow variables is defined by constraints (1i). By setting duration-related and fixed costs $\gamma = C = 0$, the system (1a)–(1i) is the basic vehicle-indexed formulation of the SDVRPTW. Desaulniers (2010) strengthens this formulation by adding tighter bounds on the fleet size, capacity cuts, and 2-path cuts. We explain these cuts later in the context of our branch-and-cut approach in Section 4.2.

Constraints (1j) and (1k) come from Property 2 and 3 respectively. They are redundant for model (1a)–(1i), but will turn out helpful in our new compact model.

Constraints (1l)–(1n) reduce or eliminate customer inconvenience caused by deferred and multiple visits. Constraints (1l) and (1m) limit the maximum number of visits to customers, individually and in total. Temporal synchronization of visits is guaranteed by constraints (1n), where $\Delta_i = 0$ imposes simultaneous deliveries and $\Delta_i = l_i - e_i$ allows to spread them arbitrarily in the service time window.

4. A Branch-and-Cut Algorithm

In this section, we extend the branch-and-cut algorithm proposed by Bianchessi and Irnich (2016) to address the SDVRPTW-IC. The algorithm is based on a compact formulation that in fact *constitutes* a relaxation of the problem. This means that some integer solutions to the relaxed formulation are infeasible for the SDVRPTW-IC. Valid inequalities are used in order to strengthen the relaxed compact formulation and possibly cut off solutions that are infeasible for the SDVRPTW-IC. However, even with the valid inequalities, integer solutions to the new compact formulation remain to be tested for feasibility. The positive arc flow values in any given integer solution to the relaxed formulation induce a subnetwork of the original instance. As there are only few split customers in a typical solution, such a subnetwork will regularly contain only few arcs. Hence, all time-window feasible routes on this subnetwork can be enumerated. An extended set-covering problem is then solved in order to decide on the selection of routes, their schedules, the quantities to deliver to the visited customers, and, hence, overall feasibility. All solutions proved infeasible are cut off from the feasible region of the relaxed problem.

In Section 4.1, we define the relaxed compact formulation for the SDVRPTW-IC and show how an optimal solution to this formulation may not be feasible to the original problem. In Section 4.2, we summarize the valid inequalities used in order to strengthen the relaxed formulation and cut off solutions that are infeasible for the SDVRPTW-IC. Finally, in Section 4.3, we present the feasibility-checking procedure and the feasibility cuts.

4.1. Relaxed Compact Formulation

The relaxed compact formulation for the SDVRPTW-IC is a two-commodity flow formulation with additional variables and constraints. The first commodity represents the available vehicles and the second represents the service times imposed by the routes. The formulation uses

(i) integer variables z_i indicating the number of times vertex $i \in N$ is visited by the vehicles;
(ii) integer flow variables x_{ij} indicating the flow of the vehicles along arc $(i,j) \in A$;
(iii) non-negative continuous flow variables T_{ij} indicating the service start time at $i \in N$ when a vehicle travels directly from i to $j \in N$; moreover, T_{0i} is the sum of the departure times at the depot 0 of the vehicles traveling along $(0,i)$, and T_{in+1} is the sum of the service start times at customer i of the vehicles traveling along $(i, n+1)$;

(iv) non-negative continuous variables w_{ij} indicating the waiting time at $j \in N$ when a vehicle travels directly from i to j for $(i,j) \in A(N)$;

(v) non-negative continuous variables E_i representing the earliest service time at customer $i \in N$.

In the remainder, we will refer to T_{ij} and w_{ij} as service-time and waiting-time flow variables respectively.

We use the following additional notation. We define $\Gamma^+_N(S) = \Gamma^+(S) \cap A(N)$ and $\Gamma^-_N(S) = \Gamma^-(S) \cap A(N)$. Again, we write $\Gamma^+_N(i)$ and $\Gamma^-_N(i)$ for singleton sets $S = \{i\}$. Finally, we define $K_S = \lceil \sum_{i \in S} d_i/Q \rceil$ as the minimum number of vehicles required to serve customers in set $S \subseteq N$.

The relaxed two-commodity flow formulation for the SDVRPTW-IC is as follows:

$$\min \quad \sum_{(i,j)\in A} c_{ij}x_{ij} + \gamma\Big(\sum_{(i,j)\in A} t_{ij}x_{ij} + \sum_{(i,j)\in A_N} w_{ij}\Big) + C \sum_{(0,i)\in\Gamma^+_N(0)} x_{0i} \tag{2a}$$

$$\text{s.t.} \quad \sum_{(h,i)\in\Gamma^-(i)} x_{hi} = \sum_{(i,j)\in\Gamma^+(i)} x_{ij} = z_i \qquad i \in N \tag{2b}$$

$$\sum_{(0,j)\in\Gamma^+(0)} x_{ij} = K \tag{2c}$$

$$\sum_{(i,j)\in\Gamma^+(S)} x_{ij} \geq K_S \qquad S \subseteq N,\ |S| \geq 2 \tag{2d}$$

$$\sum_{(h,i)\in\Gamma^-(i)} \Big(T_{hi} + t_{hi}x_{hi}\Big) + \sum_{(h,i)\in\Gamma^-_N(i)} w_{hi} = \sum_{(i,j)\in\Gamma^+(i)} T_{ij} \qquad i \in N \tag{2e}$$

$$e_i x_{ij} \leq T_{ij} \leq l_i x_{ij} \qquad (i,j) \in A \tag{2f}$$

$$\max\{0, e_j - t_{ij} - l_i\}x_{ij} \leq w_{ij} \leq \max\{0, l_j - t_{ij} - e_i\}x_{ij} \qquad (i,j) \in A(N) \tag{2g}$$

$$z_i \geq \lceil d_i/Q \rceil \text{ and integer} \qquad i \in N \tag{2h}$$

$$x_{ij} \in \{0,1\} \qquad (i,j) \in A(N) \tag{2i}$$

$$x_{ij} \geq 0 \text{ and integer} \qquad (i,j) \in A \setminus A(N) \tag{2j}$$

with customer inconvenience constraints

$$z_i \leq n_i^{\max} \qquad i \in N \tag{2k}$$

$$\sum_{i\in N} z_i \leq n^{\max} \tag{2l}$$

$$E_i \leq T_{ij} + l_i(1 - x_{ij}) \qquad (i,j) \in A(N) \tag{2m}$$

$$T_{ij} \leq E_i + \Delta_i \qquad (i,j) \in A(N) \tag{2n}$$

The objective function (2a) calls for the minimization of the total costs. Constraints (2b) impose flow conservation for the vehicle flow variables. (2c) is the fleet size constraint. Constraints (2d) prevent the generation of paths not connected to the depot. Moreover, as shown by Bianchessi and Irnich (2016), (2d) are necessary but not sufficient for maintaining capacity constraints. Constraints (2e)–(2g) impose conservation for the service-time flow, ensure consistency among the T_{ij}, w_{ij}, and x_{ij} variable values, and partially ensure time-window prescriptions. Constraints (2h)–(2j) define the domains for the integer variables. Note that the binary requirement in (2i) results from Property 2.

Constraints (2k)–(2n) are the customer inconvenience constraints. (2k) explicitly specify an upper bound on the number of visits at each customer, and (2l) enforce a limit on the overall number of deliveries performed. (2m) and (2n) are the synchronization constraints which guarantee that all visits to a customer i are performed within the time interval Δ_i.

An optimal solution to (2) may not be feasible for the SDVRPTW-IC. Bianchessi and Irnich (2016) discuss examples showing that an optimal solution to the relaxed formulation for the SDVRPTW can violate the capacity or time-window constraints. Those examples apply also to the SDVRPTW-IC. Consider the following

Example 1. *The instance depicted in Figure 2 shows that an integer solution to* (2) *can violate synchronization constraints even though it is feasible w.r.t. capacity and time-window constraints. In this instance, the depicted arcs have costs and travel times equal to 1, while all other arcs (not shown) have costs and* *travel times equal to 2. The demands d_i and the time windows $[e_i, l_i]$ of the $n = 5$ customers are presented close to each customer $i \in \{1, 2, \ldots, 5\}$. The depot time window is assumed to be non-constraining, i.e., $[e_0, l_0] = [e_{n+1}, l_{n+1}] = [0, 10]$. The capacity of the vehicles is $Q = 10$. The depicted arcs have a flow of 1 and form the unique optimal solution to the relaxed model (2). In fact, two fully loaded vehicles are required to serve the 5 customers and, due to the given customer demands, one of the customers must receive split deliveries. Therefore, the solution consists of two routes, for a total of 8 arcs. Selecting any set of arcs different from those depicted would increase the cost of the solution. As far as time-window prescriptions, demands, and vehicle capacity are concerned, this optimal solution can be converted into a feasible SDVRPTW-IC solution, e.g., using the two routes $(0, 1, 3, 4, n+1)$ and $(0, 2, 3, 5, n+1)$. In the first route, the values of the service-time flow variables T_{ij} with $i = 3$ or $j = 3$ are uniquely defined: $T_{13} = 4$ and $T_{34} = 5$. In the second route, different values are possible for the T_{ij} variables. In particular, when customers are served as early as possible, then $T_{23} = 1$ and $T_{35} = 2$. If customers are served as late at possible, then $T_{23} = 2$ and $T_{35} = 3$. If $\Delta_3 \geq 2$, then the corresponding SDVRPTW-IC solution with the as-late-as-possible schedule for the second route is feasible with regard to synchronization constraints (service times at customer 3 are then 5 and 3 and thus differ by not more than Δ_3). However, if $\Delta_3 = 1$, then customer 3 cannot be served by routes $(0, 1, 3, 4, n+1)$ and $(0, 1, 3, 5, n+1)$ in such a way that synchronization constraints are satisfied in a feasible SDVRPTW-IC solution. Nevertheless, the assignments $T_{01} = 3$, $T_{13} = 4$, $T_{34} = 4$, $T_{46} = 5$ and $T_{02} = 0$, $T_{23} = 1$, $T_{35} = 3$, $T_{56} = 4$ to the service-time flow variables are feasible for model (2).*

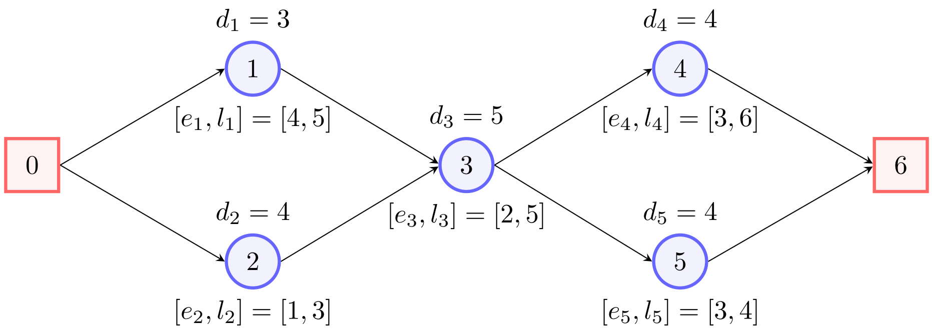


Figure 2: Optimal solution to formulation (2) that is infeasible for the SDVRPTW-IC w.r.t. synchronization constraints.

The above example has shown that the relaxed model (2) contains infeasible integer solutions w.r.t. the synchronization constraints of SDVRPTW-IC. When the minimization of the route durations becomes part of the objective, i.e., for $\gamma > 0$, model (2) also contains integer solutions that are feasible w.r.t. routing but infeasible w.r.t. scheduling. In this case, the solution represented by values of the routing variables x_{ij} can be converted into a feasible SDVRPTW-IC solution. However, such a feasible SDVRPTW-IC solution requires a different schedule than what the T_{ij} variable values indicate. In consequence, model (2) evaluates the solution given by the x_{ij} variables with a too small objective value, computed with an infeasible set of associated T_{ij} variable values.

Example 2. *An example for such a relaxed solution is presented in Figure 3. Here, the only feasible SDVRPTW-IC solution comprises the routes $(0, 1, 3, 4, n+1)$ and $(0, 2, 3, 5, n+1)$. Due to duration minimization, the values $T_{01} = 4$, $T_{13} = 5$, $T_{34} = 6$, $T_{46} = 8$, and $w_{34} = 1$ of the service-time flow and waiting time variables in the first route are unique. For the second route, different sets of values can instead be assigned to the service-time flow and waiting time variables: When customers are served as early as possible, then $T_{02} = 0$, $T_{23} = 1$, $T_{35} = 2$, and $T_{56} = 3$. In contrast, when customers are served as late at possible, then $T_{02} = 2$, $T_{23} = 3$, $T_{35} = 4$, and $T_{56} = 5$. With both schedules, the second vehicle never waits along the second route. Hence, the overall waiting time is unique and given by $w_{34} = 1$. In contrast, the values $T_{01} = 4$, $T_{13} = 5$, $T_{34} = 7$, $T_{46} = 8$, $w_{34} = 0$ and $T_{02} = 1$, $T_{23} = 2$, $T_{35} = 2$, $T_{56} = 3$ of the service-time flow and waiting variables are feasible for the relaxed model (2). Here, no waiting seems to be necessary. The objective (2a) of the relaxed model underestimates the true SDVRPTW-IC costs for the feasible x-values by $\gamma > 0$.*

Note that model (2) can be reformulated without making use of the waiting time flow variables. Objective

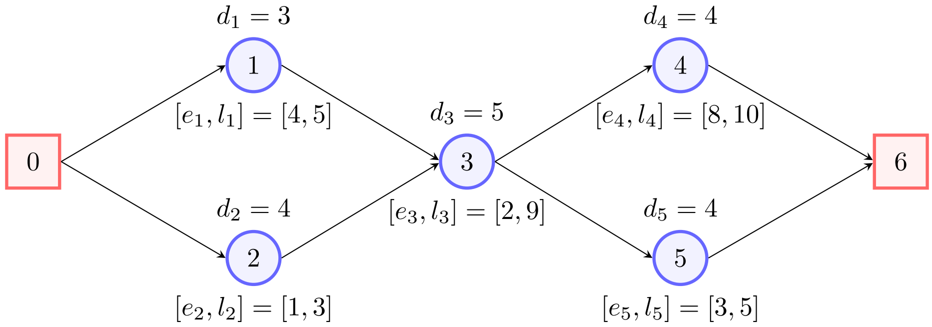


Figure 3: Optimal solution to formulation (2) in which the arc flow variables represent a set of feasible SDVRPTW-IC routes. The objective (2a) however underestimates the true route durations and costs, because optimal values for the service-time and waiting flow variables in (2) are infeasible for the routes.

(2a) and constraints (2e) and(2g) need to be replaced. The relaxed formulation becomes:

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} + \gamma \sum_{(i,j)\in A} t_{ij}x_{ij} + \gamma \sum_{i\in N}\Bigg(\sum_{(i,j)\in\Gamma^+(i)} T_{ij} - \sum_{(h,i)\in\Gamma^-(i)} \Big(T_{hi} + t_{hi}x_{hi}\Big)\Bigg) + C \sum_{(0,i)\in\Gamma^+_N(0)} x_{0i} \quad (3a)$$

$$\sum_{(h,i)\in\Gamma^-(i)} \Big(T_{hi} + t_{hi}x_{hi}\Big) \leq \sum_{(i,j)\in\Gamma^+(i)} T_{ij} \qquad i \in N \quad (3b)$$

$$\sum_{(h,i)\in\Gamma^-_N(i)} w^{LB}_{hi}x_{hi} \leq \sum_{(i,j)\in\Gamma^+(i)} T_{ij} - \sum_{(h,i)\in\Gamma^-(i)} \Big(T_{hi} + t_{hi}x_{hi}\Big) \leq w^{UB}_{hi}x_{hi} \qquad i \in N \quad (3c)$$

$$\text{(2b)–(2d), (2f), (2h)–(2n)} \quad (3d)$$

where $w^{LB}_{hi} = \max\{0, e_i - t_{hi} - l_h\}$ and $w^{UB}_{hi} = \max\{0, l_i - t_{hi} - e_h\}$. As (3c) are the aggregate form of (2g), the arising formulation is slightly weaker than (2). However, the new formulation (3) has $\mathcal{O}(n^2)$ fewer variables and constraints, and preliminary experiments showed this is beneficial from the computational point of view. Our branch-and-cut algorithm is therefore based on (3).

4.2. Valid Inequalities

In classical branch-and-cut algorithms, valid inequalities are used to strengthen the formulation of the problem addressed. Since (3) is a relaxed formulation, in our algorithm valid inequalities are also used to cut off integer solutions to (3) that are infeasible for the SDVRPTW-IC.

We consider the same classes of valid inequalities as Bianchessi and Irnich (2016):

- Inequalities

$$x_{ij} + x_{ji} \leq 1 \qquad (i,j),(j,i) \in A(N) : i < j, \quad (4)$$

  which can be imposed due to Property 3.
- Capacity cuts (2d) as stated in the previous section.
- 2-path cuts, introduced by Kohl *et al.* (1999):

$$\sum_{(i,j)\in\Gamma^+(S)} x_{ij} \geq 2, \quad (5)$$

  which apply whenever a subset $S \subseteq N$ of the customers cannot be served with a single vehicle.
- Connectivity cuts of the form

$$\sum_{(i,j)\in\Gamma^+(S)} x_{ij} \geq z_u \qquad S \subseteq N, |S| \geq 2, u \in S. \quad (6)$$

  They prove useful even though already the capacity cuts ensure that any subset of customers is connected to the depot.

- *Infeasible-path constraints* and *path-matching constraints*, introduced by Bianchessi and Irnich (2016). These are two new classes of valid inequalities for the SDVRPTW. The former are an adaptation to the SDVRPTW of the cuts bearing the same name and introduced by Ascheuer *et al.* (2000, 2001). The latter are a generalization of the former involving several partial paths starting or ending at a specified customer vertex. It is straightforward to prove that both types of cuts are also valid for the SDVRPTW-IC. Their derivation, though, is very involved and laborious and requires extensive additional notation, so that the reader is referred to the original reference for details on their definition and separation. Note, however, that by exploiting Property 5, a superset of valid infeasible-path and path-matching constraints can be defined, because this property imposes a minimum delivery amount of 1 for each visit. Bianchessi and Irnich (2016) state inequalities with closed-form expressions covering the cases with and without Property 5.

Inequalities (4) are added to the formulation right from the start, whereas the other cuts are dynamically separated in the course of the algorithm. We apply the same separation strategies as Bianchessi and Irnich (2016).

4.3. Feasibility Checking

Recall that every time a feasible integer solution to the relaxed formulation (3) is found, a procedure must check whether the solution is also feasible to the SDVRPTW-IC. If not, a feasibility cut must be inserted to cut off this solution from the feasible region of the relaxed problem.

The checking procedure we use is based on the one proposed by Bianchessi and Irnich (2016) and works as follows. Let $\bar{s} = (\bar{\mathbf{x}}, \bar{\mathbf{z}}, \bar{\mathbf{T}}, \bar{\mathbf{E}})$ be an integer solution to the relaxed formulation (3), possibly augmented by branching and cutting constraints. Let $\bar{Z} = \mathbf{c}^\top \bar{\mathbf{x}}$ denote the costs of the solution.

For $\bar{V} = V$ we define a residual network $H(\bar{V}, \bar{\mathbf{x}}) = (\bar{V}, \bar{A})$, with $\bar{A} = \{(i,j) \in A : \bar{x}_{ij} \geq 1\} \cup \{(0,j) : j \in \bar{N}\} \cup \{(i, n+1) : i \in \bar{N}\}$. Furthermore, let $\bar{S} = \{i \in \bar{N} : \bar{z}_i \geq 2\}$ be the set of customers receiving split deliveries in solution $\bar{s}$ (*split customers*). For the *non-split customers* $i \in \bar{N} \setminus \bar{S}$, we know that the delivery quantity is identical to d_i independently of the route serving the customer. Moreover, if Property 5 holds, the minimum delivery amount to split customers is equal to 1. According to these minimum delivery amounts, we define $\bar{R}$ as the set of all elementary 0-$(n+1)$-paths (routes) in $H(\bar{V}, \bar{\mathbf{x}})$ satisfying time-window and vehicle capacity constraints. We generate $\bar{R}$ by exploring $H(\bar{V}, \bar{\mathbf{x}})$ in a depth-first way.

An instance of the SDVRPTW-IC, defined on the basis of $\bar{V}$ and $\bar{\mathbf{x}}$ imposing the route set $\bar{R}$, can be modeled by a path-based formulation. Some additional notation is required. Let $\bar{N}(r) \subseteq \bar{N}$ be the subset of customers visited by route $r \in \bar{R}$ using the definition $\bar{N} = \bar{V} \setminus \{0, n+1\}$. We distinguish between routes $\bar{R}^s$ visiting a single customer, i.e., routes of the form $(0, i, n+1)$ for $i \in N$, and routes $\bar{R}^m$ visiting more than one customer. Obviously, $\bar{R} = \bar{R}^m \cup \bar{R}^s$ and $\bar{R}^m \cap \bar{R}^s = \varnothing$.

The schedule of a route needs to be feasible regarding time-window and synchronization constraints. In order to guarantee a feasible schedule for the route $r \in \bar{R}$, it suffices to impose constraints on the visit times at the vertices $i \in V_r^{time}$, where V_r^{time} is the set $(\bar{N}(r) \cap \bar{S}) \cup \{0, n+1\}$. We define the relation P_r^{time} so that $(i,j) \in P_r^{time}$ if and only if $i, j \in V_r^{time}$ and i is visited before j in route r with no other vertex of V_r^{time} in between.

Extended Set-Covering Model. The path-based formulation for the SDVRPTW-IC, defined relatively to $\bar{V}$ and $\bar{\mathbf{x}}$, uses then

(i) non-negative integer and binary variables λ^r indicating the number of vehicles assigned to route $r \in \bar{R}^s$ and $\bar{R}^m$ respectively,

(ii) non-negative continuous variables δ_i^r indicating the quantity delivered to customer $i \in \bar{N}(r) \cap \bar{S}$ by route $r \in \bar{R}$,

(iii) non-negative continuous variables T_i^r representing the service time at customer $i \in \bar{N}(r) \cap \bar{S}$, the departure time at the depot $i = 0$, and the arrival time at the depot $i = n+1$ for route $r \in \bar{R}^m$,

and it reads as follows:

$$\bar{Z}_{\bar{R}} = \min\ \gamma \sum_{r \in \bar{R}^m} \left(T_{n+1}^r - T_0^r\right) + \gamma \sum_{\substack{r \in \bar{R}^s: \\ r=(0,i,n+1)}} (t_{0i} + t_{i,n+1})\, \lambda^r + \sum_{r \in \bar{R}} (c^r + C)\, \lambda^r \tag{7a}$$

$$\text{s.t.}\quad \gamma \sum_{r \in \bar{R}^m} \left(T_{n+1}^r - T_0^r\right) + \gamma \sum_{\substack{r \in \bar{R}^s: \\ r=(0,i,n+1)}} (t_{0i} + t_{i,n+1})\, \lambda^r + \sum_{r \in \bar{R}} (c^r + C)\, \lambda^r \leq \bar{Z}^* \tag{7b}$$

$$\sum_{r \in \bar{R}: i \in \bar{N}(r)} \delta_i^r \geq d_i \qquad i \in \bar{S} \tag{7c}$$

$$\sum_{r \in \bar{R}: i \in \bar{N}(r)} \lambda_i^r \geq 1 \qquad i \in \bar{N} \setminus \bar{S} \tag{7d}$$

$$\sum_{i \in \bar{S} \cap \bar{N}(r)} \delta_i^r + \sum_{i \in (\bar{N} \setminus \bar{S}) \cap \bar{N}(r)} d_i \lambda_i^r \leq Q \lambda^r \qquad r \in \bar{R} \tag{7e}$$

$$e_i^r \lambda^r \leq T_i^r \leq l_i^r \lambda^r \qquad r \in \bar{R}^m, i \in V_r^{time} \tag{7f}$$

$$T_i^r + t_{ij}^r \lambda^r \leq T_j^r \qquad r \in \bar{R}^m, (i,j) \in P_r^{time} \tag{7g}$$

$$\sum_{r \in \bar{R}} (b_{ij}^r + b_{ji}^r) \lambda^r \leq 1 \qquad (i,j), (j,i) \in \bar{A}(\bar{N}), i < j \tag{7h}$$

$$\sum_{r \in \bar{R}} \lambda^r \leq K \tag{7i}$$

$$\delta_i^r \geq 0 \qquad r \in \bar{R}, i \in \bar{N}(r) \cap \bar{S} \tag{7j}$$

$$\lambda^r \in \{0,1\} \qquad r \in \bar{R}^m \tag{7k}$$

$$\lambda^r \geq 0 \text{ and integer} \qquad r \in \bar{R}^s \tag{7l}$$

with customer inconvenience constraints

$$\sum_{r \in \bar{R}} \sum_{(i,j) \in \Gamma^+(i)} b_{ij}^r \lambda^r \leq n_i^{\max} \qquad i \in N \tag{7m}$$

$$\sum_{r \in \bar{R}} \sum_{(i,j) \in A: i \in N} b_{ij}^r \lambda^r \leq n^{\max} \tag{7n}$$

$$E_i \leq T_i^r + l_i(1 - \lambda^r) \qquad r \in \bar{R}^m, i \in \bar{N}(r) \cap \bar{S} \tag{7o}$$

$$T_i^r \leq E_i + \Delta_i \qquad r \in \bar{R}^m, i \in \bar{N}(r) \cap \bar{S} \tag{7p}$$

where c^r are the variable routing costs of route $r \in \bar{R}$, $\bar{Z}^*$ is the upper bound to the SDVRPTW-IC stored in the branch-and-cut algorithm, t_{ij}^r is the time required to travel (without waiting) from i to j along route r, if $i, j \in \bar{N}(r) \cap \bar{S}$ and i precedes j, and b_{ij}^r is a binary arc indicator equal to 1 if arc $(i,j) \in \bar{A}(\bar{N})$ is used in route $r \in \bar{R}$, 0 otherwise.

The objective function (7a) minimizes the costs of all routes in use. If model (7) is infeasible, we set $\bar{Z}_{\bar{R}} = \infty$. Constraints (7b) impose an upper bound on the objective value $\bar{Z}_{\bar{R}}$. Constraints (7c) and (7d) ensure that customer demands are met. Vehicle capacity constraints are imposed by (7e). Constraints (7f) and (7g) define the values of the service time variables associated with split customers. Property 3 implies constraints (7h). Constraint (7i) guarantees that the fleet size is respected. Finally, constraints (7j)–(7l) define the domains of the δ_i^r and λ^r variables.

Concerning customer inconvenience constraints , (7m) and (7n) limit the maximum number of visits to customers, individually and in total, and (7o) and (7p) impose synchronization of visits.

Note that constraints (7b)–(7l) do not impose that each arc $(i,j) \in \bar{A}$ be traversed exactly $\bar{x}_{ij}$ times by the selected routes. Moreover, $\bar{A}$ may include arcs in $\Gamma^+(0) \cup \Gamma^-(n+1)$ that are not used in solution $\bar{s}$. Alternative SDVRPTW-IC solutions are thus possible, and improving solutions are found whenever $\bar{Z}_{\bar{R}} < \bar{Z}$. In addition, customer visits with zero deliveries are possible in (7), i.e., $\lambda^r > 0$ but $\delta_i^r = 0$ for some $i \in \bar{N}(r) \cap \bar{S}$. Due to the validity of the triangle inequality and because waiting is allowed at no cost, improving (or at least not worse) alternative feasible solutions can be derived by removing customers with a delivery quantity of 0 from the routes in a solution to (7). Thus, we apply a greedy postprocessing procedure in order to identify high-quality solutions as early as possible in the course of the branch-and-cut. For the sake of exposition, we assume that $\bar{Z}_{\bar{R}}$ is updated to the value of such an improving solution whenever one is detected.

If $\bar{Z}_{\bar{R}} \leq \bar{Z}$, then also $\bar{Z} \leq \bar{Z}^*$ holds, and a feasible integer solution to the SDVRPTW-IC has been found. In case $\bar{Z}_{\bar{R}} < \bar{Z}$, the solution is a new best one, so that the best known solution value can be updated by $\bar{Z}^* := \bar{Z}_{\bar{R}}$ and the branch-and-bound node can be terminated.

If $\bar{Z}_{\bar{R}} > \bar{Z}$, the current integer solution $\bar{s}$ is infeasible, and a feasibility cut must be added (see below). Moreover, the resulting branch-and-bound node must be examined further. It is worth noting that the upper bound $\bar{Z}^*$ can however be updated by $\bar{Z}^* := \bar{Z}_{\bar{R}}$ if $\bar{Z}_{\bar{R}} < \bar{Z}^*$ holds.

Feasibility Cuts. The definitions of valid feasibility cuts and the procedures to identify them are different depending on whether service-time flow variables T_{ij} occur in the objective (i.e., $\gamma > 0$ in (3a)) or not

($\gamma = 0$). The case of $\gamma = 0$ is identical to what is described in (Bianchessi and Irnich, 2016) so that can sketch it only briefly. The case of $\gamma > 0$ requires a special treatment that we describe afterwards.

If $\gamma = 0$, feasibility cuts are generated as follows. Integer solutions $\bar{s}$ to (3) often partition the set of customers into several weakly connected components. Defining $\mathcal{C}$ as the index set of these components, let $\bar{N}^c$, for each $c \in \mathcal{C}$, be the vertex set of the cth weakly connected component of $H(V, \bar{\mathbf{x}})(N)$, i.e., of the vertex-induced subgraph of $H(V, \bar{\mathbf{x}})$ induced by the customers N. Smaller SDVRPTW-IC instances can now be defined by $\bar{V}^c = \bar{N}^c \cup \{0, n+1\}$.

For each $c \in \mathcal{C}$, we define $\bar{x}^c_{ij} = \bar{x}_{ij}$ if $(i,j) \in \bar{V}^c \times \bar{V}^c$, and 0 otherwise. Then, we build $H(\bar{V}^c, \bar{\mathbf{x}}^c) = (\bar{V}^c, \bar{A}^c)$, generate the routes $\bar{R}$ over $H(\bar{V}^c, \bar{\mathbf{x}}^c)$, and solve the resulting formulation (7). Note that, in order to speed up the solution process, here we define $\bar{A}^c = \{(i,j) \in A \cap (\bar{V}^c \times \bar{V}^c) : \bar{x}_{ij} \geq 1\}$ and impose in (7) to use each arc $(i,j) \in \bar{V}^c \times \bar{V}^c$ exactly $\bar{x}^c_{ij}$ times (the additional constraints are of the form $\sum_{r \in \bar{R}} b^r_{ij} \lambda^r = \bar{x}^c_{ij}$). Moreover, we set $\bar{Z}^*$ in (7b) to $\bar{Z}^c := \mathbf{c}^\top \bar{\mathbf{x}}^c$.

If (7) is infeasible, we add the following feasibility cut defined w.r.t. the cth weakly connected component $\bar{N}^c$

$$\sum_{(i,j) \in \widehat{A}^c} x_{ij} \geq 1, \tag{8}$$

where the arc set $\widehat{A}^c$ defining the left-hand side is

$$\widehat{A}^c = \{(i,j) \in A \cap (\bar{V}^c \times \bar{V}^c) : \bar{x}_{ij} = 0\} \cup \Gamma^+_N(\bar{N}^c) \cup \Gamma^-_N(\bar{N}^c).$$

The cut (8) imposes that either the set of active vehicle flow variables associated with the internal arcs of component c must be different from the ones positive in the solution $\bar{s}$ or the component c itself must change. The inequality is globally valid. Thus, whenever $\bar{s}$ has been proved to be infeasible for the SDVRPTW-IC, it can be cut off by imposing to change the current solution for at least one connected component of $H(V, \bar{\mathbf{x}})$. It happens regularly that lifted feasibility cuts for several components can be added at the same time.

If $\gamma > 0$, i.e., if the objective contains costs related to route durations, the checking procedure outlined above is not directly applicable, as it may erroneously prevent a component $\bar{N}^c$ from being part of a solution. This is caused by the combined effect of the following: (i) the solution of the relaxed model (3) may underestimate the costs of a component (see Example 2 and Figure 3) and (ii) the feasibility cuts (8) are defined just in terms of the x_{ij} variables, which are associated with the variable routing costs only. Thus, if $\gamma > 0$, (7b) must be removed from (7) when checking the feasibility of a component. Then, a component can be proved to be infeasible due to the violation of vehicle capacity, time-window, or synchronization constraints, so that a feasibility cut (8) can be added for this component. The remaining inconvenience constraints are always satisfied, because we impose the additional constraints $\sum_{r \in \bar{R}} b^r_{ij} \lambda^r = \bar{x}^c_{ij}$ for all $(i,j) \in \bar{A}^c$ when checking the feasibility of a component. If none of the components is infeasible, the feasibility cut for checking the whole solution has to be added to the model, i.e., the feasibility cut defined for the arc set $\widehat{A}^c = \{(i,j) \in A : \bar{x}_{ij} = 0\}$.

5. Experimental Results

The branch-and-cut algorithm was implemented in C++ using CPLEX 12.6.0.1 with Concert Technology, and compiled in release mode with MS Visual C++ 2013. The experiments were performed on a 64-bit Windows 10 PC equipped with an Intel Xeon processor E5-1650v3 clocked at 3.50 GHz and with 64 GB of RAM, by allowing a single thread for each run. CPLEX's built-in cuts were used in all experiments. To improve numerical stability, we set `IloCplex::NumericalEmphasis = CPX_ON` and `IloCplex::EpGap` equal to `1.0e-5` for fixed vehicle costs $C = 0$ and to `1.0e-9` for $C = 1{,}000{,}000$ respectively. Finally, we set `IloCplex::ParallelMode = 1` in order to force CPLEX to always use deterministic algorithms. CPLEX's default values were kept for all remaining parameters.

5.1. Instances

In Section 1, we found that the standard benchmark for SDVRPTW which is based on the benchmark of Solomon (1987) lacks generality because instances do not exhibit different demand distributions. The demand distribution however strongly impacts the average savings resulting from allowing split deliveries. Therefore, we created 560 new test instances, again derived from the well-known VRPTW instances by Solomon (1987). Recall that the Solomon instance set comprises 56 instances, each of which contains 100 customers located in a 100×100 square. The set is divided into 6 classes termed R1, R2, C1, C2, RC1, and RC2, where "R" stands for "random", "C" for "clustered", and "RC" for "random and clustered", thus denoting the manner in which the customers are located in the square. The "2" instances have less constraining time windows and larger vehicle capacities than the "1" instances, so that longer routes are possible. Costs and travel times between customers are set to the Euclidean distance, customer demands are integer, and the vehicles are assumed to be homogeneous. Each class contains between 8 and 12 instances.

For the new instances, the vehicle capacity Q is set to 100. We consider five *scenarios* with regard to the customer demands:

$$D1 : [10; 70] \quad D2 : [10; 50] \quad D3 : [30; 70] \quad D4 : [30; 50] \quad D5 : [50; 70]$$

In each of the five scenarios $[a, b]$, the demand d_i of customers $i \in N$ is drawn from a discrete uniform distribution in $[\frac{a}{100}Q, \frac{b}{100}Q]$. As in the original Solomon benchmark, all instances of a class (e.g., R1) share the identical demand realization in a scenario.

From each instance, we derived 25- and 50-customer instances by considering only the first 25 and 50 customers respectively. Hence, we obtained $56 \cdot 5 \cdot 2 = 560$ instances, available at `http://logistik.bwl.uni-mainz.de/benchmarks.php`. We partitioned the instances into groups by Solomon class, demand scenario, and number of customers. For example, "C1D2N25" refers to the 25-customer instances created from Solomon class C1 with demands in $[10; 50]$.

According to the usual convention, we computed travel times and variable routing costs with one decimal place and truncation. Then, as the triangle inequality is assumed to hold for both times and costs, at preprocessing time we apply the Floyd-Warshall algorithm to times and costs independently. Hence, the new instances allow us to require all Properties 1–5 for optimal solutions.

5.2. Results

We considered the eight *distribution policies* described in Table 2. The extreme policies are those leading to the VRPTW (no splitting at all) and the SDVRPTW (arbitrary splits allowed), while the introduction of the inconvenience measures creates variants of the SDVRPTW-IC.

Table 2: The different distribution policies considered in the computational experiments

Policy	**Meaning**
VRPTW	Standard VRP with time windows.
SDVRPTW	Split delivery VRP with time windows.
SΔ, for $\Delta = 0$	SDVRPTW with temporal synchronization of deliveries/visits. $\Delta = 0$ is exact temporal synchronization.
NVν, for $\nu = 2, 3$	SDVRPTW with at most ν visits per customer, i.e., $n_i^{\max} = \nu$ for all customers $i \in N$.
TNVx, for $x = 25, 50, 75$	SDVRPTW with a limit on the total number of visits, $n^{\max}$. For an instance with n customers and ξ visits in the optimal SDVRPTW solution, $n^{\max} = n + \lceil \frac{x}{100} \cdot (\xi - n) \rceil$. Example: For an instance with $n = 50$ for which the optimal SDVRPTW solution visits ten customers twice and no customer more than twice, $\xi = 60$, and for $x = 25$, $n^{\max} = 53$.

The VRPTW served as baseline against which the other distribution policies were compared. We consider synchronization and limiting the number of visits as alternative measures for controlling inconvenience and therefore analyzed them separately; mixing them makes no sense in our opinion.

We performed three sets of experiments using different objectives (henceforth referred to as Objective I, II, and III), as defined in Table 3.

In the first set, we used the minimization of total variable routing costs. We analyzed the structure of the different solutions comparing the objective function values and the impact of the distribution policies on route durations and on the number of routes. In the second set, we included the costs related to route durations into the objective, and in the third set, we chose a hierarchical objective of minimizing the number of vehicles first (by setting very high fixed vehicle costs) and minimizing the sum of variable routing costs and costs related to route durations second. An instance was used for the analyses only when it had been solved to optimality for all policies (except NV3, as only very few instances had more than three visits in

Table 3: The different objective functions used in the computational experiments

Objective function	Objective function components: Variable routing costs	Costs related to route durations	Fixed vehicle costs
I	yes	no	no
II	yes	yes: $\gamma = 1$	no
III	yes	yes: $\gamma = 1$	yes: $C = 1{,}000{,}000$

Table 4: Effect of the different objective functions on solution structure of SDVRPTW compared to VRPTW

Objective/ Policy	Average of: Number of visits per customer	Percent-age of split customers	Number of visits per split customer	Percentage of split customers with deliveries fully synchronized	Timespan between first and last delivery in relation to time window width in %
Objective I					
SDVRPTW	1.10	9.84	2.00	20.95	29.44
NV2	1.10	9.98	2.00	21.37	28.37
S0	1.10	9.86	2.00	100.00	0.00
TNV25	1.03	3.06	2.00	22.09	24.11
TNV50	1.06	5.53	2.02	21.88	30.60
TNV75	1.08	7.84	2.00	21.35	28.17
Objective II					
SDVRPTW	1.03	2.99	2.01	10.00	13.76
NV2	1.03	2.99	2.00	10.74	14.21
S0	1.02	2.40	2.01	100.00	0.00
TNV25	1.01	0.89	2.00	15.38	8.95
TNV50	1.02	1.65	2.00	10.98	12.86
TNV75	1.02	2.37	2.00	10.00	12.85
Objective III					
SDVRPTW	1.04	3.72	2.01	19.17	17.18
NV2	1.04	3.72	2.00	11.39	19.25
S0	1.03	3.10	2.01	100.00	0.00
TNV25	1.01	1.13	2.00	15.15	8.92
TNV50	1.03	2.56	2.00	18.33	17.71
TNV75	1.03	3.20	2.00	15.00	17.32

the optimal SDVRPTW solution) and all objective functions. This was the case for 115 instances, 109 of which had 25 customers.

The results are summarized in Tables 4 and 5. Table 4 contains structural information about the effect of allowing split deliveries according to the different objective functions. It displays several KPIs that quantify how the optimal solutions of the policies with splits differ from those of the respective VRPTW. The last column deserves some explanation. If, for example, a customer with a time window of $[10, 20]$ is visited twice, at time points 13 and 16, then the "timespan between the first and the last delivery in relation to time window width" is $(16 - 13)/(20 - 10) = 0.3 = 30\%$. Note that the values in this column are based on the original time windows (as these would be given by the customers), not on the ones reduced according to the minimum arrival time from the depot and the maximum departure time to reach the depot.

Table 5 provides information on the benefits of split deliveries. The table shows the minimum, average, and maximum relative savings in % and the number of instances with savings of more than 3% for the different objective functions, each compared to the VRPTW policy with the respective objective. Note: It turned out that there are only very few instances with more than two visits to any customer, so the results for policy NV3 are omitted from the analyses.

5.2.1. Comparison of VRPTW and SDVRPTW

Looking at Table 4, one can see that the percentage of split customers depends strongly on the objective function. This also holds for the percentage of split customers for which the deliveries are fully synchronized automatically, i.e., for which all deliveries occur at the same time without requiring this by a constraint. Both values are by far highest for Objective I, i.e., when only variable routing costs are taken into account.

Table 5: Relative savings obtained with the different objective functions for SDVRPTW compared to VRPTW

Objective/ Policy	Min./Avg./Max. % Savings/# Instances with savings > 3% in				
	Objective value	Number of routes	Variable routing costs	Route durations	Sum of variable routing costs and costs related to route durations
Objective I					
SDVRPTW	0.00/2.55/8.87/47	0.00/2.25/13.33/30	0.00/2.55/8.87/47	–81.47/–7.91/16.99/8	–70.56/–5.87/15.84/8
NV2	0.00/2.55/8.87/47	0.00/2.25/13.33/30	0.00/2.55/8.87/47	–81.47/–7.92/18.46/9	–70.56/–5.87/17.19/9
S0	0.00/2.50/8.87/42	0.00/2.25/13.33/30	0.00/2.50/8.87/42	–402.52/–54.36/2.31/0	–332.27/–40.32/2.39/0
TNV25	0.00/1.30/5.49/16	–10.00/0.00/10.00/11	0.00/1.30/5.49/16	–46.82/–3.42/27.69/13	–38.40/–2.65/25.14/11
TNV50	0.00/2.15/8.19/25	0.00/2.11/13.33/28	0.00/2.15/8.19/25	–42.10/–4.81/34.26/12	–35.64/–3.62/24.15/10
TNV75	0.00/2.42/8.45/40	0.00/2.19/13.33/29	0.00/2.42/8.45/40	–56.30/–6.46/34.26/12	–47.81/–4.82/24.15/11
Objective II					
SDVRPTW	0.00/0.47/2.07/0	0.00/1.17/18.18/15	–1.15/1.03/4.86/21	–2.34/0.06/2.01/0	0.00/0.47/2.07/0
NV2	0.00/0.47/2.07/0	0.00/1.17/18.18/15	–1.15/1.02/4.86/21	–2.34/0.06/2.01/0	0.00/0.47/2.07/0
S0	0.00/0.40/2.06/0	–10.00/1.00/18.18/15	0.00/0.91/4.86/17	–2.85/0.03/1.72/0	0.00/0.40/2.06/0
TNV25	0.00/0.21/1.71/0	–10.00/–0.28/10.00/4	0.00/0.37/2.82/0	–0.97/0.09/1.31/0	0.00/0.21/1.71/0
TNV50	0.00/0.32/2.01/0	–10.00/0.43/10.00/8	–1.15/0.60/3.14/1	–1.66/0.11/2.01/0	0.00/0.32/2.01/0
TNV75	0.00/0.41/2.01/0	–10.00/0.76/18.18/12	–1.15/0.84/4.86/9	–2.34/0.10/2.01/0	0.00/0.41/2.01/0
Objective III					
SDVRPTW	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–2.50/2.43/12.71/37	–2.51/1.77/40.57/15	–2.50/2.15/36.72/17
NV2	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–2.50/2.43/12.71/37	–2.51/1.78/40.57/15	–2.50/2.15/36.72/17
S0	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–2.50/2.31/12.71/34	–3.39/1.69/40.17/15	–3.03/2.04/36.41/16
TNV25	0.00/0.08/9.07/1	0.00/0.08/9.09/1	–4.15/0.76/12.30/8	–2.41/0.74/37.93/8	–3.11/0.83/34.43/8
TNV50	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–4.15/2.01/13.72/19	–2.41/1.65/39.08/15	–3.11/1.89/35.61/16
TNV75	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–2.50/2.29/13.72/26	–2.51/1.77/39.08/15	–2.50/2.08/35.61/17

Table 5 shows that for Objective I, i.e., the minimization of variable routing costs, considerable savings in the objective value and in the number of routes are realized, averaging to 2.6 and 2.3% respectively, with reductions of more than 3% for 47 and 30 instances out of 115. Route durations, however, show a large average increase of 7.9%. What is more, the volatility of the route duration changes is high, ranging from a duration reduction of 17.0% to an increase of as much as 81.5%. As a side effect, assuming $\gamma = 1$ as for the other objectives, the sum of variable routing costs and costs related to route durations increases on average by about 5.9%. In particular, increases occur also when the number of vehicles is not reduced.

The picture changes for Objective II, i.e., when variable routing costs and costs related to route durations are minimized simultaneously. Then, the average savings in the objective function as well as in the number of routes, although still non-negligible, are much lower than for Objective I, and there is no instance with an objective reduction of more than 3%. This indicates that split deliveries pay off less when variable routing as well as duration-related costs are considered compared to the situation where only variable routing costs matter. Route durations and variable routing costs are hardly affected, and their volatility is small, with percentage savings ranging in $[-2.3, 2.0]$ and $[-1.2, 4.9]$ respectively.

For Objective III, i.e., the hierarchical objective of minimizing first the number of routes and then the sum of variable routing costs and costs related to route durations, we observe that there is only a marginal reduction in the number of routes. For the second objective function component, however, substantial savings are obtained, of 2.15% on average, and with a maximum of 36.7%. (Note that increases in the second objective function component occurred, but only when the number of vehicles was reduced.) The volatilities of the changes for variable routing costs and route durations are relevant and even higher than those found for Objective I. However, percentage savings ranges are now unbalanced towards positive values. For 17 out of 115 instances, the value of at least one of the two objective function components was reduced by at least 3%. In conclusion, it can be said that splitting pays off for Objective III, and more so than for Objective II.

5.2.2. Comparison of the Distribution Policies for the Reduction of Inconvenience

Having established the usefulness of split deliveries empirically, we evaluate in this section the different measures for reducing inconvenience that may result from splitting.

Table 4 shows that the relative values of the structural KPIs within one objective function are similar for all three of them: (i) The percentage of split customers is lower when there is a limit on the total number of visits. (ii) The percentage of fully synchronized visits and the average time span between the first and the last delivery per split customer in relation to the time window width are similar for all policies without explicit synchronization. In particular, the latter value is rather high, which may be regarded a considerable inconvenience for customers.

Looking at Table 5, the most striking observations are: (i) A limit on the overall number of visits yields, in general, smaller objective function reductions than the other measures. (ii) The NV-2 values for all columns are almost the same as for the corresponding SDVRPTW. (iii) Most notably, when costs related to route durations are ignored in the objective function, their increase is drastic for the synchronized SDVRPTW, with an average of 54.4% and a maximum of 402.5%. However, when costs related to route durations are taken into account, the duration differences between the SDVRPTW and the S0 policies are minimal. (iv) Objective function values of the SDVRPTW and the S0 policies differ only slightly for all three objectives.

As a limit on the number of individual visits does not improve the quality of service to the customers, synchronization, i.e. the S0 distribution policy, can be seen as the best measure to mitigate the customer inconvenience, leading to a win-win situation for carriers and customers.

5.2.3. In-Depth Analysis of Objective II

Objective II is important because it is the one that balances the two most critical and conflicting cost components: it simultaneously minimizes variable routing costs and costs related to route durations. In order to further validate and extend the findings stated in Sections 5.2.1 and 5.2.2, we carried out an in-depth analysis of Objective II.

Limiting the scope to Objective II, 205 instances were solved to optimality with all policies, including 18 instances with 50 customers. We obtained identical optimal SDVRPTW and VRPTW solutions for 112 of these 205 instances (identical w.r.t. to the objective function value and the number of vehicles used). In Figure 4, we display, for the remaining 93 instances and the different distribution policies, the savings achieved in total costs and number of vehicles. Information is grouped by demand scenario.

Even if cost savings are on average smaller than for Objective I as stated in Section 5.2.1, allowing split deliveries for Objective II is still a worthwhile alternative. Indeed, the magnitude of the savings very much depends on the demand distribution. Figure 4(c) reveals that, for many instances, substantial savings can be achieved, in particular in demand scenario D3.

As for the comparison of the distribution policies, the difference between NV2 and SDVRPTW is marginal:

- NV2 achieves the same cost savings as SDVRPTW in all but two cases.
- The number of vehicles used is identical for NV2 and SDVRPTW.
- NV2 is as inconvenient for customers as SDVRPTW; it reduces the number of visits only in rare cases.

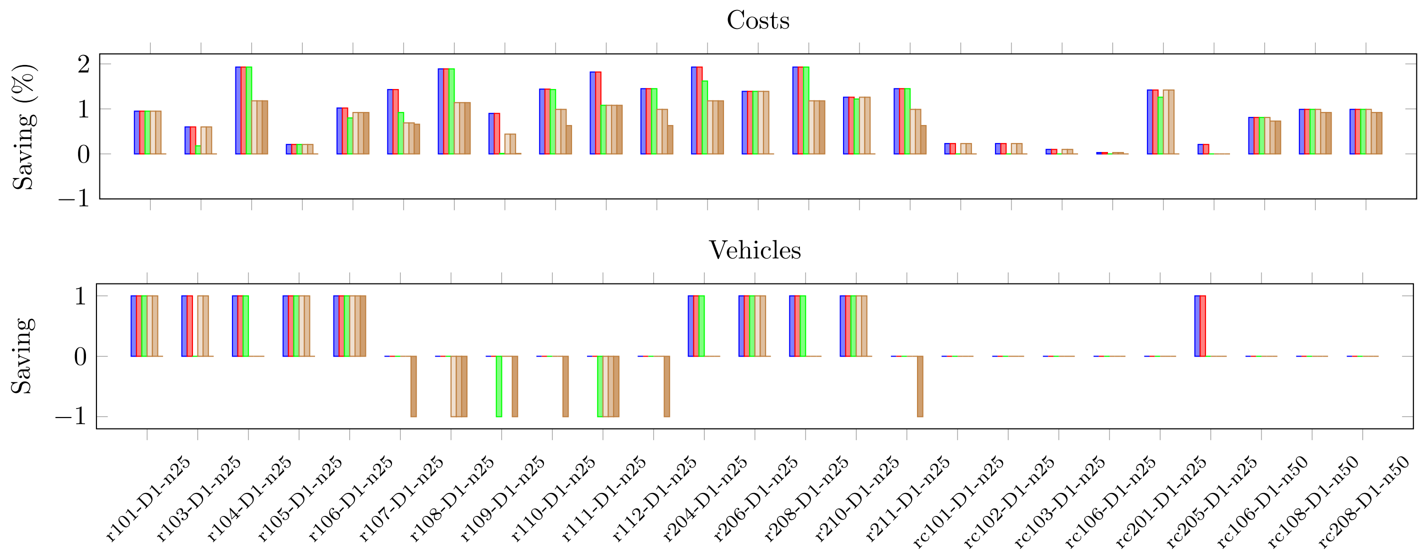
Regarding cost savings w.r.t. VRPTW, the difference in savings achieved between NV2 and S0 is greater than 0.5% (1%) in only 13 (2) out of 205 cases, with a maximum of 1.26%. Then, comparing the optimal solutions, we found that

- in 9 out of 205 cases, S0 uses 1 vehicle more than for NV2;
- in 22 (1) out of 205 cases, TNV75 uses 1 (2) vehicle(s) more than NV2;
- in 31 (11, 1) out of 205 cases, TNV50 uses 1 (2, 3) vehicle(s) more than NV2;
- in 43 (18, 1) out of 205 cases, TNV25 uses 1 (2, 5) vehicle(s) more than NV2.

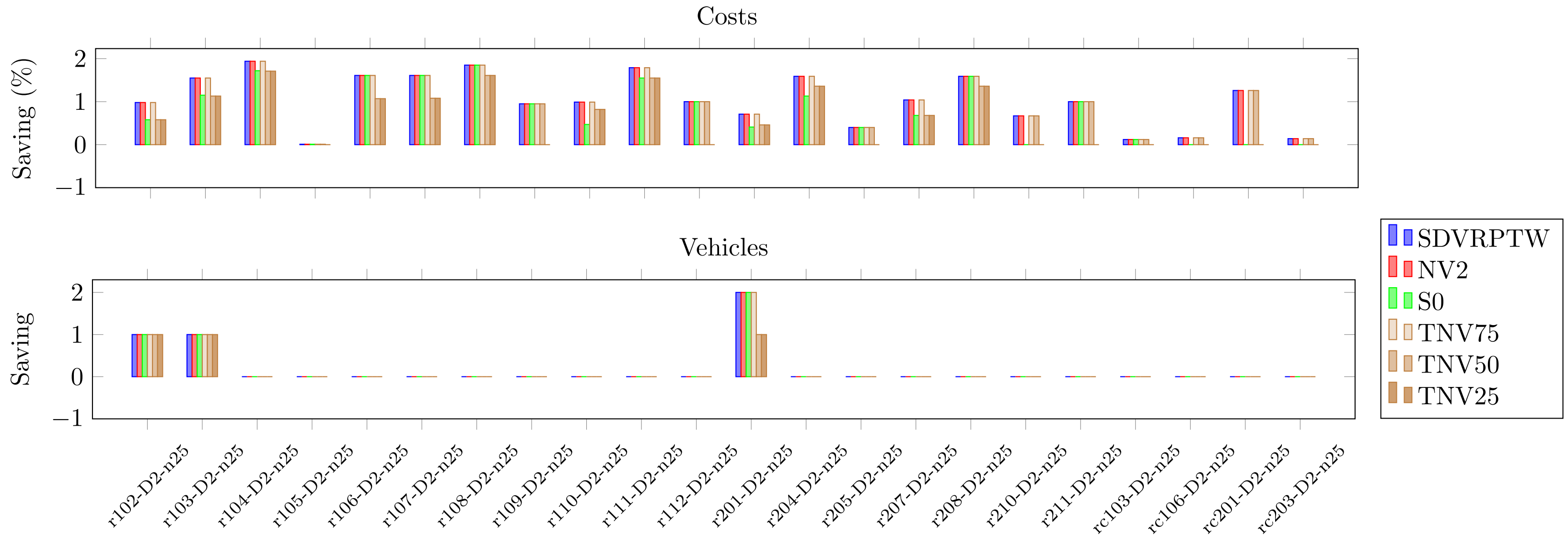
Thus, as observed in Section 5.2.2, synchronization with policy S0 is, w.r.t. total costs, the third best option after SDVRPTW and NV2. Nevertheless, S0 is superior to SDVRPTW and NV2 in reducing customer inconvenience, because in the former all visits to a customer occur at the same time.

5.2.4. Results grouped by Demand Scenario and Solomon Class

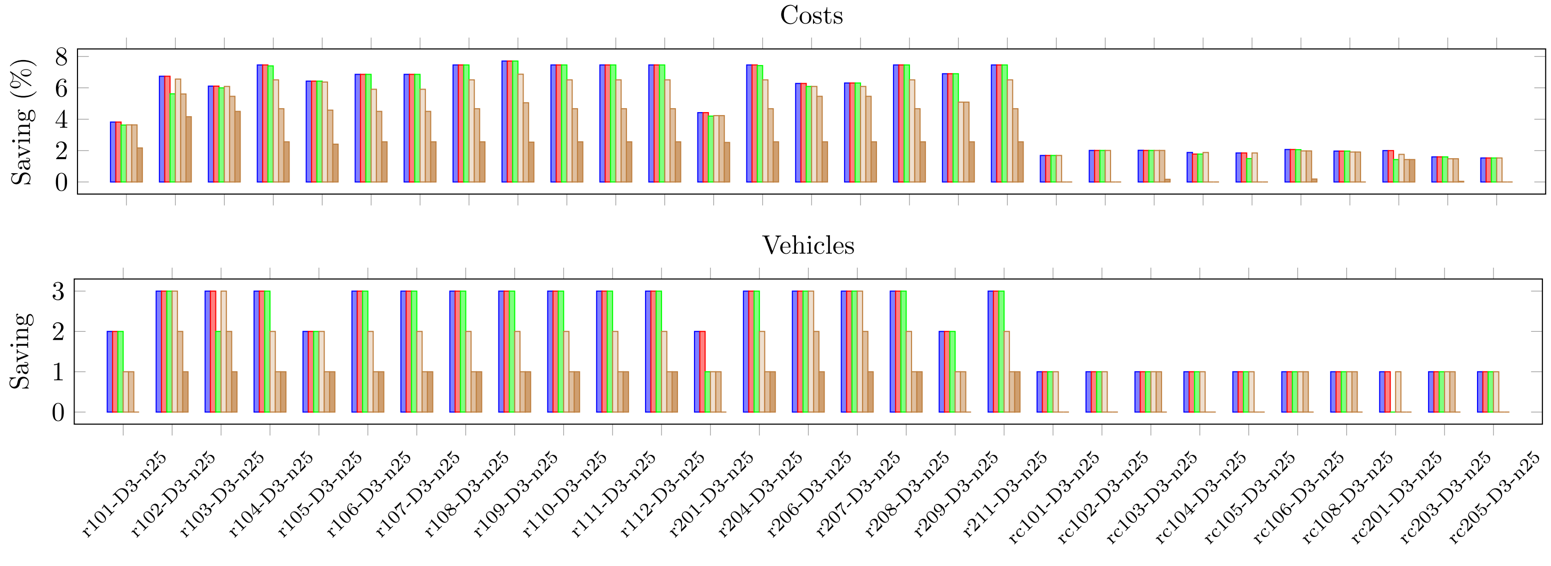
The Appendix provides further details on the aggregated results presented in Tables 4 and 5. In Section A of the Appendix, results are grouped by demand scenario. Accordingly, Tables 6–11 show the effects of the different objective functions on the solution structure of the various SDVRPTW policies compared to the VRPTW, and Tables 9–11 indicate the relative savings obtained with the different objective functions. In Section B of the Appendix, results are grouped by Solomon class. Tables 12–14 and Tables 15–17 show the respective results in this case.



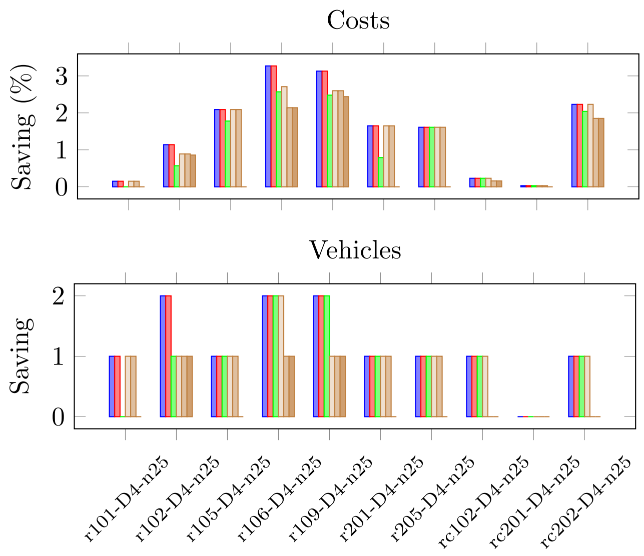
(a) Demand scenario D1



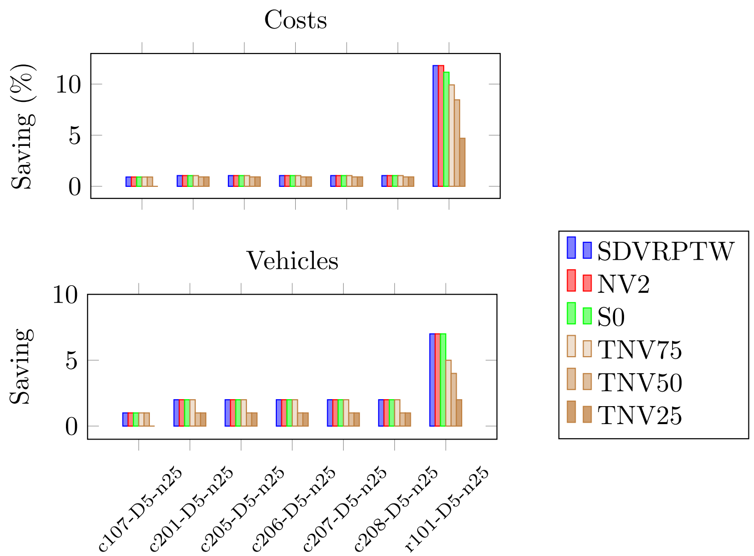
(b) Demand scenario D2



(c) Demand scenario D3



(d) Demand scenario D4



(e) Demand scenario D5

Figure 4: Savings achieved by means of the different distribution policies in case of Objective II w.r.t. the VRPTW policy

6. Conclusions

In the present paper, we have investigated the possibilities and limitations of split deliveries with the aim of creating a win-win situation for carriers and customers in goods distribution systems. It is clear that, for the customer, it is most convenient to have only one delivery per request. However, for the carrier, split deliveries offer more degrees of freedom in routing and hence a higher a higher optimization potential, i.e., more opportunities for cost reduction. A good trade-off between customer inconvenience and cost savings needs to be found. We focused our analysis on the vehicle routing problem with time windows in which split deliveries are allowed (SDVRPTW), and considered different distribution policies that either limit the number of visits to customers (individually and in total) or ensure temporally synchronized deliveries to the same customer. We evaluated the impact of these measures on carrier efficiency by means of different objective functions, each of which takes into consideration a specific combination of variable routing costs, costs related to route durations, and fixed fleet costs. The combination of these three cost components has not been considered in the literature before. We have highlighted the need to take all of them into account to provide a more complete picture of the overall logistics costs.

Based on several analyses of computational studies with a large set of instances and demand scenarios, we can make the following final recommendations to logistics managers:

- In general, split deliveries pay off; they should be considered independent of the objective.
- When variable routing costs and costs related to route durations are relevant, split deliveries are less beneficial than for other objectives, but still an alternative worth considering.
- A limit on the number of visits to individual customers is not an effective measure to mitigate customer inconvenience resulting from split deliveries, as it hardly changes the number of visits w.r.t. the SDVRPTW, i.e., it does not improve the quality of service to the customers.
- According to the average percentage of split customers, a moderate limit on the total number of visits seems to be a valid measure to reduce customer inconvenience.
- Nevertheless, the synchronization of visits allows in general to find better results. Visit synchronization, if properly implemented in practice, causes only very minor increases in any of the three components of logistics costs and therefore appears to be the most sensible and useful distribution policy.

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Appendix

A. Results grouped by Demand Scenario

The 115 instances solved to optimality with the three objective functions are divided between the different demand scenarios as follows:

D1	42
D2	51
D3	14
D4	8
D5	0

This means that scenario D5 (the one having the highest average customer demand in relation to vehicle capacity) is clearly the hardest to solve, and scenarios D3 and D4 are still considerably more difficult than D1 and D2.

The subsequent Tables 6–11 provide further details on the aggregated results given in Tables 4 and 5: Tables 6–8 show the effects of the different objective functions on the solution structure of the different SDVRPTW policies compared to the VRPTW, and Tables 9–11 indicate the relative savings obtained with the different objective functions. For obvious reasons, scenario D5 is omitted from the tables.

The following observations can be made in Tables 6–11:

- The number of visits and the percentage of split customers is highest for scenarios D3 and D4. This is to be expected, as these scenarios have a higher ratio of customer demand to vehicle capacity.
- For Objective I, the number of customers with deliveries fully synchronized is by far highest for scenario D4. For Objectives II and III, this value is, in general, highest for scenario D3 (note, however, that there are no split customers at all for scenario D4 instances with Objective II).
- For Objectives I and III, the timespan between the first and last delivery in relation to the time window width is highest for scenario D3. For Objective II, there is no discernible pattern.
- Savings are generally highest for scenario D3.

This suggests that scenario D3, i.e., a demand pattern where the average demand of a customer is between 30 and 70% of the vehicle capacity, is particularly promising for split delivery distribution strategies.

Table 6: Effect of Objective I on solution structure of SDVRPTW compared to VRPTW, grouped by demand scenario

Policy/ Demand scenario	Average of: Number of visits per customer	Percentage of split customers	Number of visits per split customer	Percentage of split customers with deliveries fully synchronized	Timespan between first and last delivery in relation to time window width in %
SDVRPTW					
D1	1.10	9.95	2.00	22.22	25.26
D2	1.06	6.00	2.00	7.71	30.77
D3	1.21	21.43	2.00	29.44	41.78
D4	1.14	13.50	2.00	65.63	21.25
Avg.	1.10	9.84	2.00	20.95	29.44
NV2					
D1	1.10	9.95	2.00	22.62	23.76
D2	1.06	6.31	2.00	8.75	29.56
D3	1.21	21.43	2.00	28.42	41.66
D4	1.14	13.50	2.00	65.63	21.73
Avg.	1.10	9.98	2.00	21.37	28.37
S0					
D1	1.10	9.95	2.00	100.00	0.00
D2	1.06	6.04	2.00	100.00	0.00
D3	1.21	21.43	2.00	100.00	0.00
D4	1.14	13.50	2.00	100.00	0.00
Avg.	1.10	9.86	2.00	100.00	0.00
TNV25					
D1	1.04	3.67	2.00	10.26	31.38
D2	1.02	1.76	2.00	28.00	10.66
D3	1.05	5.43	2.00	14.29	57.93
D4	1.04	4.00	2.00	75.00	12.57
Avg.	1.03	3.06	2.00	22.09	24.11
TNV50					
D1	1.05	5.19	2.00	16.67	34.44
D2	1.04	3.88	2.00	16.25	26.03
D3	1.12	11.14	2.14	30.36	42.98
D4	1.08	8.00	2.00	62.50	17.87
Avg.	1.06	5.53	2.02	21.88	30.60
TNV75					
D1	1.09	8.76	2.00	22.02	25.88
D2	1.05	4.59	2.00	12.08	27.20
D3	1.16	16.00	2.00	17.50	46.12
D4	1.10	9.50	2.00	70.83	14.95
Avg.	1.08	7.84	2.00	21.35	28.17

Table 7: Effect of Objective II on solution structure of SDVRPTW compared to VRPTW, grouped by demand scenario

Policy/ Demand scenario	Average of: Number of visits per customer	Percentage of split customers	Number of visits per split customer	Percentage of split customers with deliveries fully synchronized	Timespan between first and last delivery in relation to time window width in %
SDVRPTW					
D1	1.04	3.90	2.00	13.16	17.21
D2	1.03	2.51	2.00	5.00	12.26
D3	1.04	3.71	2.08	16.67	16.73
D4	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
Avg.	1.03	2.99	2.01	10.00	13.76
NV2					
D1	1.04	3.90	2.00	13.16	16.25
D2	1.03	2.51	2.00	5.00	13.88
D3	1.04	3.71	2.00	22.22	17.44
D4	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
Avg.	1.03	2.99	2.00	10.74	14.21
S0					
D1	1.03	2.95	2.00	100.00	0.00
D2	1.02	2.04	2.00	100.00	0.00
D3	1.04	3.43	2.08	100.00	0.00
D4	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
Avg.	1.02	2.40	2.01	100.00	0.00
TNV25					
D1	1.01	1.10	2.00	16.67	10.07
D2	1.01	0.94	2.00	8.33	10.40
D3	1.01	0.57	2.00	50.00	5.41
D4	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
Avg.	1.01	0.89	2.00	15.38	8.95
TNV50					
D1	1.02	2.24	2.00	18.42	13.90
D2	1.02	1.57	2.00	5.00	15.81
D3	1.01	1.14	2.00	0.00	6.39
D4	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
Avg.	1.02	1.65	2.00	10.98	12.86
TNV75					
D1	1.02	2.29	2.00	13.16	15.45
D2	1.03	2.51	2.00	5.00	12.18
D3	1.03	3.43	2.00	16.67	14.87
D4	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
Avg.	1.02	2.37	2.00	10.00	12.85

Table 8: Effect of Objective III on solution structure of SDVRPTW compared to VRPTW, grouped by demand scenario

	Average of				
Policy/ Demand scenario	**Number of visits per customer**	**Percentage of split customers**	**Number of visits per split customer**	**Percentage of split customers with deliveries fully synchronized**	**Timespan between first and last delivery in relation to time window width in %**
SDVRPTW					
D1	1.05	4.57	2.00	18.12	19.69
D2	1.03	2.67	2.00	9.09	12.24
D3	1.07	6.29	2.04	37.18	30.86
D4	1.02	1.50	2.00	25.00	11.63
Avg.	1.04	3.72	2.01	19.17	17.18
NV2					
D1	1.05	4.57	2.00	15.22	20.73
D2	1.03	2.67	2.00	6.82	11.82
D3	1.06	6.29	2.00	14.10	42.66
D4	1.02	1.50	2.00	0.00	17.88
Avg.	1.04	3.72	2.00	11.39	19.25
S0					
D1	1.04	3.52	2.00	100.00	0.00
D2	1.02	2.20	2.00	100.00	0.00
D3	1.06	6.00	2.04	100.00	0.00
D4	1.02	1.50	2.00	100.00	0.00
Avg.	1.03	3.10	2.01	100.00	0.00
TNV25					
D1	1.01	1.19	2.00	15.38	6.95
D2	1.01	0.94	2.00	8.33	8.29
D3	1.02	2.00	2.00	28.57	15.11
D4	1.01	0.50	2.00	0.00	12.49
Avg.	1.01	1.13	2.00	15.15	8.92
TNV50					
D1	1.03	3.19	2.00	17.39	18.71
D2	1.02	1.73	2.00	9.09	14.65
D3	1.05	4.57	2.00	38.46	25.73
D4	1.01	1.00	2.00	0.00	17.88
Avg.	1.03	2.56	2.00	18.33	17.71
TNV75					
D1	1.03	3.33	2.00	10.87	21.46
D2	1.03	2.67	2.00	6.82	12.19
D3	1.06	5.71	2.00	38.46	23.28
D4	1.02	1.50	2.00	0.00	17.88
Avg.	1.03	3.20	2.00	15.00	17.32

Table 9: Relative savings obtained with Objective I for SDVRPTW compared to VRPTW, grouped by demand scenario

Policy/ Demand scenario	Min./Avg./Max. % Savings/# Instances with savings > 3% in				
	Objective value	**Number of routes**	**Variable routing costs**	**Route durations**	**Sum of variable routing costs and costs related to route durations**
SDVRPTW					
D1	0.44/2.37/4.61/19	0.00/0.63/9.09/3	0.44/2.37/4.61/19	–28.33/–4.70/16.99/6	–24.58/–2.91/15.84/6
D2	0.00/1.29/3.29/6	0.00/0.65/6.67/5	0.00/1.29/3.29/6	–81.47/–4.28/8.99/2	–70.56/–3.19/7.12/2
D3	3.55/6.45/8.87/14	7.14/8.47/13.33/14	3.55/6.45/8.87/14	–47.37/–13.07/0.73/0	–28.79/–9.12/1.99/0
D4	3.39/4.70/5.23/8	10.00/10.00/10.00/8	3.39/4.70/5.23/8	–58.50/–38.91/–5.21/0	–49.71/–32.81/–4.62/0
Avg.	0.00/2.55/8.87/47	0.00/2.25/13.33/30	0.00/2.55/8.87/47	–81.47/–7.91/16.99/8	–70.56/–5.87/15.84/8
NV2					
D1	0.44/2.37/4.61/19	0.00/0.63/9.09/3	0.44/2.37/4.61/19	–28.33/–4.72/18.46/6	–24.58/–2.91/17.19/6
D2	0.00/1.29/3.29/6	0.00/0.65/6.67/5	0.00/1.29/3.29/6	–81.47/–4.41/8.99/3	–70.56/–3.32/7.12/3
D3	3.55/6.45/8.87/14	7.14/8.47/13.33/14	3.55/6.45/8.87/14	–47.37/–13.07/0.73/0	–28.79/–9.12/1.99/0
D4	3.39/4.70/5.23/8	10.00/10.00/10.00/8	3.39/4.70/5.23/8	–58.50/–38.05/1.70/0	–49.71/–32.00/1.82/0
Avg.	0.00/2.55/8.87/47	0.00/2.25/13.33/30	0.00/2.55/8.87/47	–81.47/–7.92/18.46/9	–70.56/–5.87/17.19/9
S0					
D1	0.44/2.29/4.00/17	0.00/0.63/9.09/3	0.44/2.29/4.00/17	–273.23/–54.18/2.31/0	–233.31/–37.72/2.39/0
D2	0.00/1.25/3.13/3	0.00/0.65/6.67/5	0.00/1.25/3.13/3	–350.49/–50.33/0.00/0	–300.42/–37.84/0.00/0
D3	3.55/6.45/8.87/14	7.14/8.47/13.33/14	3.55/6.45/8.87/14	–119.93/–44.79/–16.20/0	–101.25/–33.40/–7.64/0
D4	3.39/4.70/5.23/8	10.00/10.00/10.00/8	3.39/4.70/5.23/8	–402.52/–97.72/–8.19/0	–332.27/–81.92/–7.40/0
Avg.	0.00/2.50/8.87/42	0.00/2.25/13.33/30	0.00/2.50/8.87/42	–402.52/–54.36/2.31/0	–332.27/–40.32/2.39/0
TNV25					
D1	0.00/1.28/3.50/5	–10.00/–2.38/0.00/0	0.00/1.28/3.50/5	–17.66/–1.26/24.48/5	–15.28/–0.83/18.49/5
D2	0.00/0.65/2.00/0	0.00/0.00/0.00/0	0.00/0.65/2.00/0	–34.49/–1.90/7.39/3	–27.45/–1.47/5.94/1
D3	0.23/2.68/5.49/5	0.00/1.43/6.67/3	0.23/2.68/5.49/5	–16.42/3.16/27.69/5	–13.85/3.59/25.14/5
D4	1.63/3.12/3.57/6	10.00/10.00/10.00/8	1.63/3.12/3.57/6	–46.82/–35.97/–22.61/0	–38.40/–30.61/–19.15/0
Avg.	0.00/1.30/5.49/16	–10.00/0.00/10.00/11	0.00/1.30/5.49/16	–46.82/–3.42/27.69/13	–38.40/–2.65/25.14/11
TNV50					
D1	0.37/1.82/3.73/7	0.00/0.41/9.09/2	0.37/1.82/3.73/7	–17.66/–1.89/18.46/5	–15.28/–0.87/17.19/5
D2	0.00/1.10/2.70/0	0.00/0.52/6.67/4	0.00/1.10/2.70/0	–34.49/–1.67/34.26/7	–27.45/–1.39/24.15/5
D3	2.99/5.83/8.19/11	7.14/8.47/13.33/14	2.99/5.83/8.19/11	–28.65/–11.65/0.95/0	–18.42/–8.50/1.86/0
D4	2.70/4.15/4.59/7	10.00/10.00/10.00/8	2.70/4.15/4.59/7	–42.10/–28.17/–6.89/0	–35.64/–23.66/–6.23/0
Avg.	0.00/2.15/8.19/25	0.00/2.11/13.33/28	0.00/2.15/8.19/25	–42.10/–4.81/34.26/12	–35.64/–3.62/24.15/10
TNV75					
D1	0.44/2.31/4.41/19	0.00/0.63/9.09/3	0.44/2.31/4.41/19	–28.33/–3.94/18.46/6	–24.58/–2.47/17.19/6
D2	0.00/1.19/2.77/0	0.00/0.52/6.67/4	0.00/1.19/2.77/0	–31.76/–2.29/34.26/6	–27.45/–1.74/24.15/5
D3	3.40/6.16/8.45/14	7.14/8.47/13.33/14	3.40/6.16/8.45/14	–47.55/–14.83/0.55/0	–28.96/–10.70/1.82/0
D4	2.70/4.34/4.97/7	10.00/10.00/10.00/8	2.70/4.34/4.97/7	–56.30/–31.57/–6.89/0	–47.81/–26.55/–6.23/0
Avg.	0.00/2.42/8.45/40	0.00/2.19/13.33/29	0.00/2.42/8.45/40	–56.30/–6.46/34.26/12	–47.81/–4.82/24.15/11

Table 10: Relative savings obtained with Objective II for SDVRPTW compared to VRPTW, grouped by demand scenario

Policy/ Demand scenario	Min./Avg./Max. % Savings/# Instances with savings > 3% in				
	Objective value	**Number of routes**	**Variable routing costs**	**Route durations**	**Sum of variable routing costs and costs related to route durations**
SDVRPTW					
D1	0.00/0.51/1.93/0	0.00/1.28/9.09/6	–1.15/1.16/4.34/11	–1.38/0.03/2.01/0	0.00/0.51/1.93/0
D2	0.00/0.42/1.94/0	0.00/0.75/18.18/3	0.00/1.01/4.86/7	–2.34/–0.01/0.95/0	0.00/0.42/1.94/0
D3	0.00/0.80/2.07/0	0.00/3.06/7.14/6	0.00/1.30/3.60/3	0.00/0.38/1.16/0	0.00/0.80/2.07/0
D4	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
Avg.	0.00/0.47/2.07/0	0.00/1.17/18.18/15	–1.15/1.03/4.86/21	–2.34/0.06/2.01/0	0.00/0.47/2.07/0
NV2					
D1	0.00/0.51/1.93/0	0.00/1.28/9.09/6	–1.15/1.16/4.34/11	–1.38/0.03/2.01/0	0.00/0.51/1.93/0
D2	0.00/0.42/1.94/0	0.00/0.75/18.18/3	0.00/1.01/4.86/7	–2.34/–0.01/0.95/0	0.00/0.42/1.94/0
D3	0.00/0.80/2.07/0	0.00/3.06/7.14/6	0.00/1.27/3.18/3	0.00/0.41/1.16/0	0.00/0.80/2.07/0
D4	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
Avg.	0.00/0.47/2.07/0	0.00/1.17/18.18/15	–1.15/1.02/4.86/21	–2.34/0.06/2.01/0	0.00/0.47/2.07/0
S0					
D1	0.00/0.43/1.93/0	–10.00/0.80/9.09/6	0.00/1.00/4.34/8	–1.19/0.00/1.72/0	0.00/0.43/1.93/0
D2	0.00/0.34/1.85/0	0.00/0.75/18.18/3	0.00/0.87/4.86/6	–2.85/–0.04/0.95/0	0.00/0.34/1.85/0
D3	0.00/0.77/2.06/0	0.00/3.06/7.14/6	0.00/1.27/3.18/3	0.00/0.36/1.16/0	0.00/0.77/2.06/0
D4	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
Avg.	0.00/0.40/2.06/0	–10.00/1.00/18.18/15	0.00/0.91/4.86/17	–2.85/0.03/1.72/0	0.00/0.40/2.06/0
TNV25					
D1	0.00/0.24/1.18/0	–10.00/–1.45/9.09/1	0.00/0.46/2.08/0	–0.97/0.08/0.92/0	0.00/0.24/1.18/0
D2	0.00/0.26/1.71/0	0.00/0.57/10.00/3	0.00/0.45/2.82/0	–0.08/0.13/1.31/0	0.00/0.26/1.71/0
D3	0.00/0.03/0.19/0	0.00/0.00/0.00/0	0.00/0.06/0.64/0	–0.21/0.00/0.20/0	0.00/0.03/0.19/0
D4	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
Avg.	0.00/0.21/1.71/0	–10.00/–0.28/10.00/4	0.00/0.37/2.82/0	–0.97/0.09/1.31/0	0.00/0.21/1.71/0
TNV50					
D1	0.00/0.35/1.42/0	–10.00/0.16/9.09/3	–1.15/0.66/2.56/0	–1.12/0.13/2.01/0	0.00/0.35/1.42/0
D2	0.00/0.34/1.71/0	0.00/0.57/10.00/3	0.00/0.71/2.82/0	–1.66/0.08/1.31/0	0.00/0.34/1.71/0
D3	0.00/0.28/2.01/0	0.00/1.02/7.14/2	0.00/0.41/3.14/1	0.00/0.18/1.45/0	0.00/0.28/2.01/0
D4	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
Avg.	0.00/0.32/2.01/0	–10.00/0.43/10.00/8	–1.15/0.60/3.14/1	–1.66/0.11/2.01/0	0.00/0.32/2.01/0
TNV75					
D1	0.00/0.36/1.42/0	–10.00/0.16/9.09/3	–1.15/0.67/2.56/0	–1.12/0.13/2.01/0	0.00/0.36/1.42/0
D2	0.00/0.42/1.94/0	0.00/0.75/18.18/3	0.00/1.01/4.86/7	–2.34/–0.01/0.95/0	0.00/0.42/1.94/0
D3	0.00/0.79/2.01/0	0.00/3.06/7.14/6	0.00/1.23/3.14/2	0.00/0.43/1.45/0	0.00/0.79/2.01/0
D4	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
Avg.	0.00/0.41/2.01/0	–10.00/0.76/18.18/12	–1.15/0.84/4.86/9	–2.34/0.10/2.01/0	0.00/0.41/2.01/0

Table 11: Relative savings obtained with Objective III for SDVRPTW compared to VRPTW, grouped by demand scenario

Policy/ Demand scenario	Min./Avg./Max. % Savings/# Instances with savings > 3% in				
	Objective value	**Number of routes**	**Variable routing costs**	**Route durations**	**Sum of variable routing costs and costs related to route durations**
SDVRPTW					
D1	0.00/0.22/9.07/1	0.00/0.22/9.09/1	–2.50/1.87/11.78/15	–2.51/0.70/17.71/2	–2.50/1.18/15.76/3
D2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/1.30/9.37/9	–1.66/0.05/1.41/0	0.00/0.49/3.12/1
D3	0.00/0.04/0.19/0	0.00/0.00/0.00/0	0.00/8.90/12.71/12	–0.67/10.86/40.57/12	0.00/10.98/36.72/12
D4	0.00/0.01/0.10/0	0.00/0.00/0.00/0	0.00/1.29/8.48/1	–0.20/2.53/20.48/1	0.00/2.36/18.78/1
Avg.	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–2.50/2.43/12.71/37	–2.51/1.77/40.57/15	–2.50/2.15/36.72/17
NV2					
D1	0.00/0.22/9.07/1	0.00/0.22/9.09/1	–2.50/1.87/11.78/15	–2.51/0.70/17.71/2	–2.50/1.18/15.76/3
D2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/1.30/9.37/9	–1.66/0.05/1.41/0	0.00/0.49/3.12/1
D3	0.00/0.04/0.19/0	0.00/0.00/0.00/0	0.00/8.87/12.71/12	–0.67/10.88/40.57/12	0.00/10.98/36.72/12
D4	0.00/0.01/0.10/0	0.00/0.00/0.00/0	0.00/1.29/8.48/1	–0.20/2.53/20.48/1	0.00/2.36/18.78/1
Avg.	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–2.50/2.43/12.71/37	–2.51/1.78/40.57/15	–2.50/2.15/36.72/17
S0					
D1	0.00/0.22/9.07/1	0.00/0.22/9.09/1	–2.50/1.72/11.78/13	–3.39/0.62/17.71/2	–3.03/1.07/15.76/3
D2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/1.16/9.37/8	–1.66/0.01/0.95/0	0.00/0.42/2.83/0
D3	0.00/0.04/0.19/0	0.00/0.00/0.00/0	0.00/8.83/12.71/12	–0.71/10.81/40.17/12	0.00/10.92/36.41/12
D4	0.00/0.01/0.09/0	0.00/0.00/0.00/0	0.00/1.29/8.48/1	–0.20/2.07/16.79/1	0.00/1.97/15.62/1
Avg.	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–2.50/2.31/12.71/34	–3.39/1.69/40.17/15	–3.03/2.04/36.41/16
TNV25					
D1	0.00/0.22/9.07/1	0.00/0.22/9.09/1	–4.15/0.22/1.91/0	–2.41/0.04/2.03/0	–3.11/0.12/1.54/0
D2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/0.54/4.47/1	0.00/0.16/1.58/0	0.00/0.32/2.79/0
D3	0.00/0.02/0.18/0	0.00/0.00/0.00/0	0.00/3.40/12.30/7	0.00/4.54/37.93/7	0.00/4.60/34.43/7
D4	0.00/0.01/0.05/0	0.00/0.00/0.00/0	0.00/0.33/2.62/0	0.00/1.38/11.02/1	0.00/1.23/9.83/1
Avg.	0.00/0.08/9.07/1	0.00/0.08/9.09/1	–4.15/0.76/12.30/8	–2.41/0.74/37.93/8	–3.11/0.83/34.43/8
TNV50					
D1	0.00/0.22/9.07/1	0.00/0.22/9.09/1	–4.15/1.42/11.78/4	–2.41/0.72/17.71/2	–3.11/1.00/15.76/3
D2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/1.05/9.37/3	–1.66/0.09/1.58/0	0.00/0.42/2.79/0
D3	0.00/0.04/0.18/0	0.00/0.00/0.00/0	0.00/8.10/13.72/12	–0.67/10.32/39.08/12	0.00/10.26/35.61/12
D4	0.00/0.01/0.05/0	0.00/0.00/0.00/0	0.00/0.56/2.62/0	–0.20/1.35/11.02/1	0.00/1.24/9.83/1
Avg.	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–4.15/2.01/13.72/19	–2.41/1.65/39.08/15	–3.11/1.89/35.61/16
TNV75					
D1	0.00/0.22/9.07/1	0.00/0.22/9.09/1	–2.50/1.47/11.78/4	–2.51/0.72/17.71/2	–2.50/1.01/15.76/3
D2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/1.30/9.37/9	–1.66/0.05/1.41/0	0.00/0.49/3.12/1
D3	0.00/0.04/0.18/0	0.00/0.00/0.00/0	0.00/8.93/13.72/12	–0.67/10.80/39.08/12	0.00/10.90/35.61/12
D4	0.00/0.01/0.10/0	0.00/0.00/0.00/0	0.00/1.29/8.48/1	–0.20/2.53/20.48/1	0.00/2.36/18.78/1
Avg.	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–2.50/2.29/13.72/26	–2.51/1.77/39.08/15	–2.50/2.08/35.61/17

B. Results grouped by Solomon Class

Similar to the demand scenarios, the difficulty of the instances we constructed varies strongly between the different Solomon classes. The 115 instances solved to optimality with all objective functions are divided between the classes as follows:

C1	27
C2	21
R1	22
R2	9
RC1	25
RC2	11

As can be seen, the instances derived from classes R2 and RC2, i.e., the random as well as the random and clustered instances with wide time windows and large vehicle capacities, are on average much harder to solve than those from other classes.

Tables 12–17 provide further details on the aggregated results given in Tables 4 and 5: Tables 12–14 show the effects of the different objective functions on the solution structure of the different SDVRPTW policies compared to the VRPTW, and Tables 15–17 indicate the relative savings obtained with the different objective functions.

The following observations can be made in the tables:

- For Objective I, the number of visits per customer and the percentage of split customers is lowest for the RC instances.
- For Objective II, there is no split customer for the C instances (there were several instances with split customers, but these were not solved to optimality for all policies).
- For Objective III, the number of visits per customer and the percentage of split customers is lower for the C and for the RC instances than for the R instances.
- The percentage of split customers with deliveries fully synchronized is lowest for the R instances.
- In general, the timespan between the first and last delivery in relation to the time window width is highest for the R1 instances.
- For Objective I, the savings in the objective function are highest for the C2 instances, as are the savings in the number of routes.
- For Objective II, both types of savings are highest for the R instances.
- For Objective III, the savings in the objective function are highest for the C2 and R1 instances; savings in the number of routes are obtained only for R1 instances.
- Solomon classes can be ranked by decreasing savings as follows: C, R, and RC for Objective I; R, RC, and C for Objective II; R1 and C2 for Objective III.

We may conclude that the effects of the geographical distribution of customers depend on the objective function.

Table 12: Effect of Objective I on solution structure of SDVRPTW compared to VRPTW, grouped by Solomon class

Policy/ Solomon class	Average of: Number of visits per customer	Percentage of split customers	Number of visits per split customer	Percentage of split customers with deliveries fully synchronized	Timespan between first and last delivery in relation to time window width in %
SDVRPTW					
C1	1.09	9.33	2.00	35.99	26.29
C2	1.13	13.14	2.00	25.34	31.28
R1	1.12	11.64	2.00	5.95	40.08
R2	1.12	11.56	2.00	2.78	25.09
RC1	1.07	7.20	2.00	19.91	26.94
RC2	1.06	5.82	2.00	20.83	21.58
Avg.	1.10	9.84	2.00	20.95	29.44
NV2					
C1	1.10	9.48	2.00	37.84	22.66
C2	1.13	12.95	2.00	25.45	31.22
R1	1.12	11.82	2.00	5.95	37.60
R2	1.13	12.89	2.00	2.78	28.40
RC1	1.07	7.12	2.00	15.74	26.42
RC2	1.06	6.00	2.00	29.17	22.90
Avg.	1.10	9.98	2.00	21.37	28.37
S0					
C1	1.10	9.48	2.00	100.00	0.00
C2	1.13	13.33	2.00	100.00	0.00
R1	1.11	11.09	2.00	100.00	0.00
R2	1.12	12.00	2.00	100.00	0.00
RC1	1.07	7.12	2.00	100.00	0.00
RC2	1.06	6.18	2.00	100.00	0.00
Avg.	1.10	9.86	2.00	100.00	0.00
TNV25					
C1	1.03	2.81	2.00	42.11	26.49
C2	1.04	3.81	2.00	40.00	11.96
R1	1.04	3.45	2.00	15.79	20.46
R2	1.04	4.00	2.00	11.11	19.21
RC1	1.02	2.40	2.00	0.00	35.40
RC2	1.02	2.18	2.00	14.29	27.15
Avg.	1.03	3.06	2.00	22.09	24.11
TNV50					
C1	1.06	5.78	2.00	35.19	36.66
C2	1.08	7.62	2.00	25.00	24.48
R1	1.07	6.55	2.00	11.90	33.82
R2	1.06	6.22	2.00	11.11	23.34
RC1	1.04	3.36	2.11	11.11	32.47
RC2	1.03	3.27	2.00	31.25	22.67
Avg.	1.06	5.53	2.02	21.88	30.60
TNV75					
C1	1.08	7.85	2.00	38.58	22.83
C2	1.10	9.71	2.00	27.94	26.05
R1	1.09	8.91	2.00	7.14	38.74
R2	1.09	9.33	2.00	0.00	29.40
RC1	1.06	6.00	2.00	11.57	28.94
RC2	1.05	5.09	2.00	29.17	21.42
Avg.	1.08	7.84	2.00	21.35	28.17

Table 13: Effect of Objective II on solution structure of SDVRPTW compared to VRPTW, grouped by Solomon class

Policy/ Solomon class	Average of				
	Number of visits per customer	**Percentage of split customers**	**Number of visits per split customer**	**Percentage of split customers with deliveries fully synchronized**	**Timespan between first and last delivery in relation to time window width in %**
SDVRPTW					
C1	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
C2	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
R1	1.08	8.18	2.00	12.70	30.19
R2	1.08	8.44	2.00	3.70	31.36
RC1	1.03	2.88	2.04	8.33	21.26
RC2	1.02	1.45	2.00	16.67	9.51
Avg.	1.03	2.99	2.01	10.00	13.76
NV2					
C1	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
C2	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
R1	1.08	8.18	2.00	12.70	28.94
R2	1.08	8.44	2.00	3.70	35.01
RC1	1.03	2.88	2.00	11.11	23.69
RC2	1.02	1.45	2.00	16.67	8.23
Avg.	1.03	2.99	2.00	10.74	14.21
S0					
C1	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
C2	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
R1	1.07	6.73	2.00	100.00	0.00
R2	1.08	7.56	2.00	100.00	0.00
RC1	1.02	1.92	2.07	100.00	0.00
RC2	1.01	1.09	2.00	100.00	0.00
Avg.	1.02	2.40	2.01	100.00	0.00
TNV25					
C1	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
C2	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
R1	1.03	2.91	2.00	12.50	30.68
R2	1.03	3.11	2.00	0.00	30.90
RC1	1.00	0.40	2.00	66.67	3.03
RC2	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
Avg.	1.01	0.89	2.00	15.38	8.95
TNV05					
C1	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
C2	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
R1	1.05	4.55	2.00	14.29	30.30
R2	1.04	4.44	2.00	5.56	34.43
RC1	1.02	1.68	2.00	11.11	17.17
RC2	1.01	0.73	2.00	0.00	6.70
Avg.	1.02	1.65	2.00	10.98	12.86
TNV75					
C1	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
C2	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
R1	1.06	6.00	2.00	14.29	24.25
R2	1.06	6.22	2.00	5.56	28.66
RC1	1.03	2.72	2.00	4.17	24.36
RC2	1.02	1.45	2.00	16.67	7.08
Avg.	1.02	2.37	2.00	10.00	12.85

Table 14: Effect of Objective III on solution structure of SDVRPTW compared to VRPTW, grouped by Solomon class

Policy/ Solomon class	Average of: Number of visits per customer	Percentage of split customers	Number of visits per split customer	Percentage of split customers with deliveries fully synchronized	Timespan between first and last delivery in relation to time window width in %
SDVRPTW					
C1	1.01	0.74	2.00	20.00	5.37
C2	1.03	2.48	2.00	55.00	10.78
R1	1.08	8.36	2.00	8.73	33.66
R2	1.08	8.44	2.00	3.70	26.09
RC1	1.03	2.88	2.04	19.44	18.45
RC2	1.02	2.18	2.00	16.67	15.30
Avg.	1.04	3.72	2.01	19.17	17.18
NV2					
C1	1.01	0.74	2.00	20.00	5.37
C2	1.03	2.48	2.00	30.00	17.22
R1	1.08	8.36	2.00	4.76	34.00
R2	1.08	8.44	2.00	0.00	30.60
RC1	1.03	2.88	2.00	11.11	22.40
RC2	1.02	2.18	2.00	16.67	11.29
Avg.	1.04	3.72	2.00	11.39	19.25
S0					
C1	1.01	0.74	2.00	100.00	0.00
C2	1.03	2.48	2.00	100.00	0.00
R1	1.07	7.09	2.00	100.00	0.00
R2	1.08	7.56	2.00	100.00	0.00
RC1	1.02	1.92	2.07	100.00	0.00
RC2	1.01	1.09	2.00	100.00	0.00
Avg.	1.03	3.10	2.01	100.00	0.00
TNV25					
C1	1.00	0.00	*n.a.*	*n.a.*	*n.a.*
C2	1.00	0.38	2.00	50.00	4.76
R1	1.03	2.91	2.00	12.50	24.52
R2	1.03	3.11	2.00	0.00	14.97
RC1	1.01	0.88	2.00	33.33	5.99
RC2	1.01	0.73	2.00	0.00	9.29
Avg.	1.01	1.13	2.00	15.15	8.92
TNV50					
C1	1.01	0.74	2.00	20.00	5.37
C2	1.02	2.10	2.00	55.00	8.83
R1	1.05	5.09	2.00	9.52	35.16
R2	1.05	5.33	2.00	0.00	30.52
RC1	1.02	2.16	2.00	20.83	18.57
RC2	1.02	1.45	2.00	0.00	17.61
Avg.	1.03	2.56	2.00	18.33	17.71
TNV75					
C1	1.01	0.74	2.00	20.00	5.37
C2	1.02	2.29	2.00	55.00	8.83
R1	1.07	6.73	2.00	4.76	31.85
R2	1.07	7.11	2.00	0.00	30.24
RC1	1.03	2.72	2.00	8.33	22.10
RC2	1.02	1.82	2.00	16.67	12.38
Avg.	1.03	3.20	2.00	15.00	17.32

Table 15: Relative savings obtained with Objective I for SDVRPTW compared to VRPTW, grouped by Solomon class

Policy/ Solomon class	Min./Avg./Max. % Savings/# Instances with savings > 3% in				
	Objective value	**Number of routes**	**Variable routing costs**	**Route durations**	**Sum of variable routing costs and costs related to route durations**
SDVRPTW					
C1	1.03/3.03/8.87/9	0.00/3.70/13.33/9	1.03/3.03/8.87/9	–58.50/–19.69/4.27/1	–49.71/–16.70/3.92/1
C2	0.76/3.90/8.51/14	0.00/2.65/10.00/7	0.76/3.90/8.51/14	–81.47/–9.29/16.99/4	–70.56/–7.45/15.84/4
R1	0.00/3.08/4.61/13	0.00/1.21/9.09/3	0.00/3.08/4.61/13	–6.99/–1.38/2.61/0	–2.65/0.48/2.82/0
R2	1.40/3.10/4.00/5	0.00/0.00/0.00/0	1.40/3.10/4.00/5	–23.21/–6.88/8.99/2	–17.73/–3.70/7.12/2
RC1	0.00/1.06/3.56/5	0.00/2.23/7.14/8	0.00/1.06/3.56/5	–6.05/–1.30/0.73/0	–3.10/–0.30/1.99/0
RC2	0.00/0.65/3.56/1	0.00/1.86/7.14/3	0.00/0.65/3.56/1	–47.37/–5.31/10.07/1	–28.79/–3.40/6.50/1
Avg.	0.00/2.55/8.87/47	0.00/2.25/13.33/30	0.00/2.55/8.87/47	–81.47/–7.91/16.99/8	–70.56/–5.87/15.84/8
NV2					
C1	1.03/3.03/8.87/9	0.00/3.70/13.33/9	1.03/3.03/8.87/9	–58.50/–20.04/4.27/1	–49.71/–17.01/3.92/1
C2	0.76/3.90/8.51/14	0.00/2.65/10.00/7	0.76/3.90/8.51/14	–81.47/–8.89/18.46/4	–70.56/–7.08/17.19/4
R1	0.00/3.08/4.61/13	0.00/1.21/9.09/3	0.00/3.08/4.61/13	–6.99/–1.18/3.96/1	–2.65/0.60/3.55/1
R2	1.40/3.10/4.00/5	0.00/0.00/0.00/0	1.40/3.10/4.00/5	–23.99/–7.19/8.99/2	–17.73/–3.90/7.12/2
RC1	0.00/1.06/3.56/5	0.00/2.23/7.14/8	0.00/1.06/3.56/5	–6.05/–1.29/0.73/0	–3.10/–0.29/1.99/0
RC2	0.00/0.65/3.56/1	0.00/1.86/7.14/3	0.00/0.65/3.56/1	–47.37/–5.42/10.07/1	–28.79/–3.47/6.50/1
Avg.	0.00/2.55/8.87/47	0.00/2.25/13.33/30	0.00/2.55/8.87/47	–81.47/–7.92/18.46/9	–70.56/–5.87/17.19/9
S0					
C1	1.03/3.03/8.87/9	0.00/3.70/13.33/9	1.03/3.03/8.87/9	–68.38/–40.74/–11.80/0	–58.96/–35.11/–10.69/0
C2	0.76/3.90/8.51/14	0.00/2.65/10.00/7	0.76/3.90/8.51/14	–402.52/–107.18/2.31/0	–332.27/–91.43/2.39/0
R1	0.00/2.87/4.00/9	0.00/1.21/9.09/3	0.00/2.87/4.00/9	–43.38/–24.70/0.00/0	–24.90/–13.36/0.00/0
R2	1.40/3.06/4.00/4	0.00/0.00/0.00/0	1.40/3.06/4.00/4	–240.19/–142.72/–58.02/0	–136.52/–90.71/–45.93/0
RC1	0.00/1.06/3.56/5	0.00/2.23/7.14/8	0.00/1.06/3.56/5	–30.26/–13.04/0.00/0	–18.30/–7.09/0.00/0
RC2	0.00/0.65/3.56/1	0.00/1.86/7.14/3	0.00/0.65/3.56/1	–119.68/–67.85/0.00/0	–74.45/–43.75/0.00/0
Avg.	0.00/2.50/8.87/42	0.00/2.25/13.33/30	0.00/2.50/8.87/42	–402.52/–54.36/2.31/0	–332.27/–40.32/2.39/0
TNV25					
C1	0.00/1.58/3.57/6	0.00/2.96/10.00/9	0.00/1.58/3.57/6	–41.94/–13.72/3.41/1	–35.72/–11.65/3.13/1
C2	0.00/2.40/5.49/10	0.00/0.95/10.00/2	0.00/2.40/5.49/10	–46.82/0.24/27.69/7	–38.40/0.55/25.14/7
R1	0.00/1.49/2.67/0	–10.00/–3.18/0.00/0	0.00/1.49/2.67/0	–6.05/0.32/3.89/2	–2.45/0.80/2.95/0
R2	1.26/1.71/2.00/0	–10.00/–3.33/0.00/0	1.26/1.71/2.00/0	–34.49/1.58/24.48/3	–27.22/1.03/16.21/3
RC1	0.00/0.24/1.33/0	0.00/0.00/0.00/0	0.00/0.24/1.33/0	–3.06/–0.76/0.60/0	–1.65/–0.34/0.92/0
RC2	0.00/0.18/0.44/0	0.00/0.00/0.00/0	0.00/0.18/0.44/0	–16.42/–2.75/0.09/0	–10.34/–1.80/0.08/0
Avg.	0.00/1.30/5.49/16	–10.00/0.00/10.00/11	0.00/1.30/5.49/16	–46.82/–3.42/27.69/13	–38.40/–2.65/25.14/11
TNV50					
C1	0.92/2.78/8.19/9	0.00/3.70/13.33/9	0.92/2.78/8.19/9	–42.10/–15.16/3.41/1	–35.64/–12.82/3.13/1
C2	0.66/3.60/7.85/13	0.00/2.65/10.00/7	0.66/3.60/7.85/13	–31.76/–5.35/18.46/4	–27.45/–4.11/17.19/4
R1	0.00/2.27/2.89/0	0.00/0.79/9.09/2	0.00/2.27/2.89/0	–6.05/0.39/6.56/5	–2.45/1.17/4.96/3
R2	1.26/2.24/2.70/0	0.00/0.00/0.00/0	1.26/2.24/2.70/0	–34.49/2.26/34.26/2	–27.22/2.00/24.15/2
RC1	0.00/0.85/3.00/2	0.00/1.96/7.14/7	0.00/0.85/3.00/2	–3.06/–0.66/0.95/0	–1.65/–0.02/1.86/0
RC2	0.00/0.50/3.03/1	0.00/1.86/7.14/3	0.00/0.50/3.03/1	–28.65/–4.01/0.00/0	–17.10/–2.48/0.00/0
Avg.	0.00/2.15/8.19/25	0.00/2.11/13.33/28	0.00/2.15/8.19/25	–42.10/–4.81/34.26/12	–35.64/–3.62/24.15/10
TNV75					
C1	1.03/2.91/8.45/9	0.00/3.70/13.33/9	1.03/2.91/8.45/9	–56.30/–18.89/4.27/1	–47.81/–16.04/3.92/1
C2	0.66/3.73/8.17/13	0.00/2.65/10.00/7	0.66/3.73/8.17/13	–31.76/–6.58/18.46/4	–27.45/–5.16/17.19/4
R1	0.00/2.88/4.41/9	0.00/1.21/9.09/3	0.00/2.88/4.41/9	–6.99/0.26/6.56/4	–2.65/1.35/4.96/3
R2	1.40/2.90/3.80/3	0.00/0.00/0.00/0	1.40/2.90/3.80/3	–23.21/–1.02/34.26/2	–17.73/0.26/24.15/2
RC1	0.00/1.02/3.41/5	0.00/1.96/7.14/7	0.00/1.02/3.41/5	–5.49/–1.21/0.55/0	–3.10/–0.26/1.82/0
RC2	0.00/0.63/3.41/1	0.00/1.86/7.14/3	0.00/0.63/3.41/1	–47.55/–5.49/8.56/1	–28.96/–3.52/5.57/1
Avg.	0.00/2.42/8.45/40	0.00/2.19/13.33/29	0.00/2.42/8.45/40	–56.30/–6.46/34.26/12	–47.81/–4.82/24.15/11

Table 16: Relative savings obtained with Objective II for SDVRPTW compared to VRPTW, grouped by Solomon class

Policy/ Solomon class	Min./Avg./Max. % Savings/# Instances with savings > 3% in				
	Objective value	**Number of routes**	**Variable routing costs**	**Route durations**	**Sum of variable routing costs and costs related to route durations**
SDVRPTW					
C1	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
C2	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
R1	0.00/1.24/1.94/0	0.00/2.53/10.00/6	0.00/2.93/4.34/12	–1.66/0.01/0.95/0	0.00/1.24/1.94/0
R2	0.67/1.32/1.93/0	0.00/4.04/18.18/3	1.59/3.41/4.86/6	–2.34/–0.20/0.29/0	0.67/1.32/1.93/0
RC1	0.00/0.46/2.07/0	0.00/1.43/7.14/5	–1.15/0.75/3.60/3	–0.80/0.23/1.27/0	0.00/0.46/2.07/0
RC2	0.00/0.28/1.53/0	0.00/0.65/7.14/1	0.00/0.38/2.60/0	–0.43/0.20/2.01/0	0.00/0.28/1.53/0
Avg.	0.00/0.47/2.07/0	0.00/1.17/18.18/15	–1.15/1.03/4.86/21	–2.34/0.06/2.01/0	0.00/0.47/2.07/0
NV2					
C1	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
C2	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
R1	0.00/1.24/1.94/0	0.00/2.53/10.00/6	0.00/2.93/4.34/12	–1.66/0.01/0.95/0	0.00/1.24/1.94/0
R2	0.67/1.32/1.93/0	0.00/4.04/18.18/3	1.59/3.41/4.86/6	–2.34/–0.20/0.29/0	0.67/1.32/1.93/0
RC1	0.00/0.46/2.07/0	0.00/1.43/7.14/5	–1.15/0.73/3.18/3	–0.80/0.24/1.27/0	0.00/0.46/2.07/0
RC2	0.00/0.28/1.53/0	0.00/0.65/7.14/1	0.00/0.38/2.60/0	–0.43/0.20/2.01/0	0.00/0.28/1.53/0
Avg.	0.00/0.47/2.07/0	0.00/1.17/18.18/15	–1.15/1.02/4.86/21	–2.34/0.06/2.01/0	0.00/0.47/2.07/0
S0					
C1	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
C2	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
R1	0.00/1.05/1.93/0	–10.00/1.62/10.00/6	0.00/2.49/4.34/8	–1.66/0.01/0.95/0	0.00/1.05/1.93/0
R2	0.00/1.09/1.93/0	0.00/4.04/18.18/3	0.00/3.10/4.86/6	–2.85/–0.38/0.29/0	0.00/1.09/1.93/0
RC1	0.00/0.41/2.06/0	0.00/1.43/7.14/5	0.00/0.73/3.18/3	–0.80/0.16/1.16/0	0.00/0.41/2.06/0
RC2	0.00/0.25/1.53/0	0.00/0.65/7.14/1	0.00/0.30/2.60/0	0.00/0.22/1.72/0	0.00/0.25/1.53/0
Avg.	0.00/0.40/2.06/0	–10.00/1.00/18.18/15	0.00/0.91/4.86/17	–2.85/0.03/1.72/0	0.00/0.40/2.06/0
TNV25					
C1	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
C2	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
R1	0.00/0.72/1.71/0	–10.00/–1.40/10.00/3	0.00/1.26/2.82/0	–0.97/0.33/1.31/0	0.00/0.72/1.71/0
R2	0.00/0.76/1.36/0	–10.00/–0.10/9.09/1	0.00/1.47/2.41/0	–0.08/0.25/0.60/0	0.00/0.76/1.36/0
RC1	0.00/0.05/0.92/0	0.00/0.00/0.00/0	0.00/0.09/1.33/0	–0.21/0.02/0.60/0	0.00/0.05/0.92/0
RC2	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
Avg.	0.00/0.21/1.71/0	–10.00/–0.28/10.00/4	0.00/0.37/2.82/0	–0.97/0.09/1.31/0	0.00/0.21/1.71/0
TNV50					
C1	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
C2	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
R1	0.00/0.91/1.71/0	–10.00/1.21/10.00/5	0.00/1.88/2.82/0	–1.66/0.21/1.31/0	0.00/0.91/1.71/0
R2	0.46/0.99/1.36/0	0.00/1.01/9.09/1	1.19/1.99/2.56/0	–0.18/0.26/0.60/0	0.46/0.99/1.36/0
RC1	0.00/0.23/2.01/0	0.00/0.57/7.14/2	–1.15/0.34/3.14/1	–0.80/0.15/1.45/0	0.00/0.23/2.01/0
RC2	0.00/0.14/1.42/0	0.00/0.00/0.00/0	0.00/0.15/0.94/0	–0.43/0.14/2.01/0	0.00/0.14/1.42/0
Avg.	0.00/0.32/2.01/0	–10.00/0.43/10.00/8	–1.15/0.60/3.14/1	–1.66/0.11/2.01/0	0.00/0.32/2.01/0
TNV75					
C1	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
C2	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
R1	0.00/1.04/1.94/0	–10.00/1.21/10.00/5	0.00/2.26/3.89/4	–1.66/0.16/0.95/0	0.00/1.04/1.94/0
R2	0.67/1.11/1.59/0	0.00/2.02/18.18/1	1.59/2.77/4.86/3	–2.34/–0.10/0.52/0	0.67/1.11/1.59/0
RC1	0.00/0.46/2.01/0	0.00/1.43/7.14/5	–1.15/0.71/3.14/2	–0.80/0.25/1.45/0	0.00/0.46/2.01/0
RC2	0.00/0.28/1.53/0	0.00/0.65/7.14/1	0.00/0.38/2.60/0	–0.43/0.20/2.01/0	0.00/0.28/1.53/0
Avg.	0.00/0.41/2.01/0	–10.00/0.76/18.18/12	–1.15/0.84/4.86/9	–2.34/0.10/2.01/0	0.00/0.41/2.01/0

Table 17: Relative savings obtained with Objective III for SDVRPTW compared to VRPTW, grouped by Solomon class

Policy/ Solomon class	Min./Avg./Max. % Savings/# Instances with savings > 3% in				
	Objective value	**Number of routes**	**Variable routing costs**	**Route durations**	**Sum of variable routing costs and costs related to route durations**
SDVRPTW					
C1	0.00/0.00/0.02/0	0.00/0.00/0.00/0	0.00/0.92/9.43/3	–1.15/0.23/8.13/1	0.00/0.35/6.79/1
C2	0.00/0.03/0.19/0	0.00/0.00/0.00/0	0.00/4.90/12.71/10	–0.39/7.54/40.57/8	0.00/7.17/36.72/9
R1	0.00/0.41/9.07/1	0.00/0.41/9.09/1	–2.50/2.78/4.45/12	–2.51/–0.04/0.95/0	–2.50/1.14/2.02/0
R2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	1.59/3.51/5.52/6	–0.29/0.23/1.41/0	0.67/1.61/3.12/1
RC1	0.00/0.00/0.02/0	0.00/0.00/0.00/0	–1.15/1.94/9.57/5	–0.80/1.22/6.38/5	0.00/1.54/7.82/5
RC2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/0.98/8.29/1	–0.43/0.70/5.45/1	0.00/0.83/6.73/1
Avg.	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–2.50/2.43/12.71/37	–2.51/1.77/40.57/15	–2.50/2.15/36.72/17
NV2					
C1	0.00/0.00/0.02/0	0.00/0.00/0.00/0	0.00/0.92/9.43/3	–1.15/0.23/8.13/1	0.00/0.35/6.79/1
C2	0.00/0.03/0.19/0	0.00/0.00/0.00/0	0.00/4.90/12.71/10	–0.39/7.54/40.57/8	0.00/7.17/36.72/9
R1	0.00/0.41/9.07/1	0.00/0.41/9.09/1	–2.50/2.78/4.45/12	–2.51/–0.04/0.95/0	–2.50/1.14/2.02/0
R2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	1.59/3.51/5.52/6	–0.29/0.23/1.41/0	0.67/1.61/3.12/1
RC1	0.00/0.00/0.02/0	0.00/0.00/0.00/0	–1.15/1.92/9.57/5	–0.80/1.24/6.38/5	0.00/1.54/7.82/5
RC2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/0.98/8.29/1	–0.43/0.70/5.45/1	0.00/0.83/6.73/1
Avg.	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–2.50/2.43/12.71/37	–2.51/1.78/40.57/15	–2.50/2.15/36.72/17
S0					
C1	0.00/0.00/0.02/0	0.00/0.00/0.00/0	0.00/0.90/9.37/3	–1.15/0.23/8.13/1	0.00/0.34/6.79/1
C2	0.00/0.03/0.19/0	0.00/0.00/0.00/0	0.00/4.91/12.71/10	–0.39/7.35/40.17/8	0.00/7.01/36.41/9
R1	0.00/0.41/9.07/1	0.00/0.41/9.09/1	–2.50/2.36/4.45/9	–3.39/–0.11/1.16/0	–3.03/0.93/2.02/0
R2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/3.20/5.52/6	–0.55/0.06/0.91/0	0.00/1.38/2.83/0
RC1	0.00/0.00/0.02/0	0.00/0.00/0.00/0	0.00/1.92/9.57/5	–0.80/1.16/6.38/5	0.00/1.50/7.82/5
RC2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/0.82/8.29/1	0.00/0.65/5.45/1	0.00/0.73/6.73/1
Avg.	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–2.50/2.31/12.71/34	–3.39/1.69/40.17/15	–3.03/2.04/36.41/16
TNV25					
C1	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0	0.00/0.00/0.00/0
C2	0.00/0.01/0.18/0	0.00/0.00/0.00/0	0.00/0.71/12.30/1	0.00/2.33/37.93/2	0.00/2.11/34.43/2
R1	0.00/0.41/9.07/1	0.00/0.41/9.09/1	–4.15/0.90/2.82/0	–2.41/0.21/1.31/0	–3.11/0.50/1.71/0
R2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/1.65/4.47/1	–0.43/0.29/1.58/0	0.00/0.86/2.79/0
RC1	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/1.24/7.04/5	0.00/0.89/5.11/5	0.00/1.05/5.99/5
RC2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/0.58/5.60/1	0.00/0.54/3.93/1	0.00/0.56/4.68/1
Avg.	0.00/0.08/9.07/1	0.00/0.08/9.09/1	–4.15/0.76/12.30/8	–2.41/0.74/37.93/8	–3.11/0.83/34.43/8
TNV50					
C1	0.00/0.00/0.02/0	0.00/0.00/0.00/0	0.00/0.92/9.43/3	–1.15/0.23/8.13/1	0.00/0.35/6.79/1
C2	0.00/0.03/0.18/0	0.00/0.00/0.00/0	0.00/4.68/13.72/9	–0.39/7.02/39.08/8	0.00/6.69/35.61/9
R1	0.00/0.41/9.07/1	0.00/0.41/9.09/1	–4.15/1.74/2.82/0	–2.41/0.07/1.31/0	–3.11/0.77/1.71/0
R2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	1.59/2.52/4.47/1	–0.18/0.33/1.58/0	0.67/1.25/2.79/0
RC1	0.00/0.00/0.01/0	0.00/0.00/0.00/0	–1.15/1.53/8.78/5	–0.80/1.05/6.13/5	0.00/1.26/7.18/5
RC2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/0.78/5.60/1	–0.43/0.52/3.93/1	0.00/0.63/4.68/1
Avg.	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–4.15/2.01/13.72/19	–2.41/1.65/39.08/15	–3.11/1.89/35.61/16
TNV75					
C1	0.00/0.00/0.02/0	0.00/0.00/0.00/0	0.00/0.92/9.43/3	–1.15/0.23/8.13/1	0.00/0.35/6.79/1
C2	0.00/0.03/0.18/0	0.00/0.00/0.00/0	0.00/4.96/13.72/10	–0.39/7.47/39.08/8	0.00/7.12/35.61/9
R1	0.00/0.41/9.07/1	0.00/0.41/9.09/1	–2.50/2.19/3.89/4	–2.51/0.02/1.27/0	–2.50/0.93/1.94/0
R2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	1.59/3.01/5.52/3	–0.22/0.21/1.41/0	0.67/1.38/3.12/1
RC1	0.00/0.00/0.02/0	0.00/0.00/0.00/0	–1.15/1.90/9.57/5	–0.80/1.25/6.38/5	0.00/1.54/7.82/5
RC2	0.00/0.00/0.01/0	0.00/0.00/0.00/0	0.00/1.03/8.29/1	–0.43/0.66/5.45/1	0.00/0.82/6.73/1
Avg.	0.00/0.09/9.07/1	0.00/0.08/9.09/1	–2.50/2.29/13.72/26	–2.51/1.77/39.08/15	–2.50/2.08/35.61/17