Optimal Booking Control in Airline Revenue Management with Two Flights and Flexible Products

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Abstract

Chen *et al.* (Chen S, Gallego G, Li M Z, and Lin B (2010) Optimal seat allocation for two-flight problems with a flexible demand segment, EJOR, 201(3), 897–908) analyze the structure of optimal booking control in the airline revenue management problem with two flights and customers that can be served by allocating a seat on either flight. Their model requires at most one request in a booking period. They derive an optimal switching curve-based policy by exploiting concavity, submodularity, and subconcavity. We address a couple of open questions with the following contributions: First, we set up a model covering a broader class of this problem. Particularly, our model applies to static demand distributions, i.e, when customers arrive in batches but can be partially accepted and assigned. We show that the monotonicity properties are valid under both dynamic and static demand models providing self-contained proofs. Second, we provide a unifying characterization of the structure of optimal booking control in the form of "booking paths". This concept formalizes the idea that an optimal allocation of a batch demand decomposes into a sequence of optimal single-request allocations. Third, we examine the relationship between booking paths and switching curves showing that both characterize equivalent policies. Computationally, this equivalence implies that there is no advantage of implementing switching curves. Rather, one can resort to the simple criteria which we propose in order to construct the optimal booking paths.

Key words: dynamic programming, airline revenue management, flexible and opaque products, optimal booking policies

1. Introduction

Airlines commonly enhance revenues by offering heterogeneous customers various fare classes for a seat in the same cabin. They typically sell the limited seat inventory over a booking horizon and demand arrives randomly. In the standard airline revenue management problem, the airline wishes to control ticket sales so as to maximize the expected revenue.

A traditional airline product combines a fare class with a seat on a flight between a particular origin-destination pair at a particular departure time in order to address customers demanding that leg. In contrast to these *specific products*, airlines and other service providers have recently been observed combining fare classes with a predefined set of alternatives. Gallego and Phillips (2004)

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define a set of alternatives serving the same market as *flexible products*. For instance, consider an airline that offers two specific products: a morning-flight and an afternoon-flight. Suppose that passengers pay more for flying in the morning than in the afternoon. If some passengers are indifferent to the departure time provided that they pay the same price as for the respective constituting specific product, the airline could launch a flexible product comprising both flights in order to capture these flexible passengers.

Moreover, other passengers might have a lower valuation if a product conceals the departure time. In this case, the airline can sell the product at a discount to attract the more price-sensitive customers but without cannibalizing too much demand for the high-fare specific products. These *opaque products* are a popular tool for price differentiation in the airline and hotel industry, especially among internet-based distributors, e.g., Priceline and Hotwire (Gallego and Phillips, 2004). Sometimes airline tickets have other hidden characteristics such as connecting flights, transfers, and the operating airline's brand name (Post, 2010; Gönsch and Steinhardt, 2013).

Our paper has a methodological focus on the structure of optimal booking control. From this point of view, the paper of Chen *et al.* (2010) is related. Chen *et al.* analyze an airline revenue management problem with two flights, specific, and flexible products, i.e., the airline needs to control both acceptance and assignment of the flexible passengers. Their model assumes the standard *dynamic demand model* (e.g., Lee and Hersh, 1993) in which at most one request arrives in a booking period. They derive an optimal switching curve-based policy by exploiting concavity, submodularity, and subconcavity. No such results are known for the well-known *static demand model* (e.g., Curry, 1990; Wollmer, 1992; Brumelle and McGill, 1993; Robinson, 1995; Li and Oum, 2002; Aydın *et al.*, 2013), i.e., when customers arrive in batches at a time but can be partially accepted.

In our paper, we adopt a unifying perspective by setting up an extended *omnibus model* with two flights, specific, flexible, and opaque products. Originally, the omnibus model introduced by Lautenbacher and Stidham (1999) marries models for the single-leg revenue management problem with dynamic and static demand. It permits the identification of the common structure underlying optimal policies under the different demand models.

Our research intends to take this line of research one step further by making the following contributions: First, our paper provides a completely self-contained analysis of important monotonicity properties of the value function (concavity, submodularity, and subconcavity) in the extended omnibus model. Our results hold particularly for the static model for which we use a novel reformulation of the value function which allows to reuse results known from the dynamic model. We also skip proving redundant properties (concavity) by focusing on multimodularity. Second, we provide a unifying characterization of the structure of optimal booking control exploiting the proven monotonicity. To this, we introduce *booking paths* to formalize the idea that an optimal allocation of a batch demand decomposes into a sequence of optimal single-request allocations and we provide a numerical example for illustration. Third, we examine the relationship between booking paths and switching curves showing that both characterizations are equivalent.

The remainder of this paper is organized as follows: In Section 2, we formulate the extended omnibus model. Section 3 introduces equivalent reformulations of the value function on which we base our analysis of the various monotonicity properties in Section 4. In Section 5, we derive a unifying booking path-based description of an optimal booking policy and show that Chen *et al.*'s switching curve-based description results in the same optimal acceptance and assignment regions. Finally, a numerical example is presented before final conclusions are drawn in Section 6.

Type of product	Constituting flight(s)	Revenues	
		r_j^1	r_j^2
Specific	1	>0	$=0$
	2	$=0$	>0
Flexible	$\{1,2\}$	>0	>0

Table 1: Possible product definitions

2. Unifying problem formulation

In this section, we set up the extended omnibus model with two flights and multiple specific, flexible, and opaque products. For the remainder of the paper, we include opaque products in our definition of flexible products as in Gallego and Phillips (2004) as the only formal difference is the choice of the fare values relative to their constituting specific products.

2.1. Notation and demand models

Throughout, $\mathbb{Z}_+$ denotes the set of nonnegative integers. Suppose that an airline operates two flight legs $i \in \mathcal{I} = \{1,2\}$ and offers a set of products $\mathcal{J} = \{1,\ldots,n\}$. A product is defined as a combination of one or two constituting flights and a fare class. A constituting flight of a product is a flight $i \in \mathcal{I}$ to which requests may be assigned and it is given by the definition of that product. Specific products have one and flexible products have two constituting flights.

Our approach to distinguish between these types of products formally is the following: For every $j \in \mathcal{J}$, we define a revenue vector denoted by $r_j = (r_j^1, r_j^2)^\top$. The revenue r_j^1 accrues to the airline if a request for product j is accepted and assigned to flight 1, and r_j^2 is accrued when we accept and assign to flight 2. Revenues associated with assigning product j customers to a constituting flight are assumed to be given as positive real values. We define the revenue of assigning a customer to a non-constituting flight of the requested product to be zero. For the flexible products, we do not assume that one or both fare values are presented to the customer before the purchase because this decision is not relevant to our model. Table 1 summarizes the product definitions that are possible in the model by properly adjusting r_j .

We assume a finite time horizon $\mathcal{T} = \{1,\ldots,\tau\}$ with $t=\tau$ being the first and $t=1$ being the last period in which bookings can be made. We let $d \in \mathbb{Z}_+$ denote the size of a batch request or simply the *demand level*. A positive demand level $d \in \{1,2,\ldots\}$ occurs with probability p_{jdt} for product $j \in \mathcal{J}$ at time $t \in \mathcal{T}$. In any period $t \in \mathcal{T}$, $d=0$ may happen if product $j \in \mathcal{J}$ is not requested, i.e., $p_{j0t} > 0$. The probabilities are assumed to be independent across time and products with

$$\sum_{j\in\mathcal{J}}\sum_{d=0}^{\infty} p_{jdt} = 1 \qquad \forall t \in \mathcal{T}.$$

We consider two classes of distributions. First, assuming that requests for different products may overlap in any period and that at most one customer arrives at a time, a distribution satisfying

$$\sum_{j\in\mathcal{J}}\sum_{d=0}^{1} p_{jdt} = 1 \qquad \forall t \in \mathcal{T} \tag{1}$$

is called a *dynamic demand distribution.* In this case, it follows that $p_{jdt} = 0$ for $d \geq 2, j \in \mathcal{J}, t \in \mathcal{T}$. We also refer to (1) as *single-request model.* Single-request models require to forecast time-specific arrival probabilities for each product, but the total demand depends implicitly on the number of booking periods τ.

Second, assuming that requests for different products are not allowed to arrive in the same booking period and letting $\mathbb{1}\{\cdot\}$ denote the indicator function, a distribution that satisfies

$$\sum_{j\in\mathcal{J}} \mathbb{1}\left\{\sum_{d=0}^{\infty} p_{jdt} > 0\right\} = 1 \qquad \forall t \in \mathcal{T} \tag{2}$$

is called *static demand distribution.* We also refer to (2) as *batch-request model.* Note that $\tau = n$ because batch-request models assume an exact pairing between products and periods. The unique product $j \in \mathcal{J}$ with $\sum_{d=0}^{\infty} p_{jdt} > 0$ in period $t \in \mathcal{T}$ is referred to as $j = j(t)$. Batch-request models require to forecast the distribution of the total demand for each product.

2.2. The extended omnibus model

The system state $x = (x_i)_{i\in\mathcal{I}}$ at any time is described by the current booking level with x_i being the number of reservations currently accepted and assigned to resource i. Let $c = (c_i)_{i\in\mathcal{I}}$ denote the initial capacities. The state space is defined by $\mathcal{X} = \mathbb{Z}_+^2$. Using nonnegative integer vectors $u_t = (u_{it})_{i\in\mathcal{I}}$ for $t \in \mathcal{T}$, we decide on the number u_{it} of observed requests to be accepted and assigned to resource i in period t. Adopting the standard partial fulfillment assumption, we define the set of feasible actions conditional on the demand level $d \in \mathbb{Z}_+$ as $\mathcal{U}(d) = \{u \in \mathbb{Z}_+^2 : u_1 + u_2 \leq d\}$.

Let $V_t(x)$ denote the maximum expected revenue from periods $t, t-1, \ldots, 0$ when there are x reservations on hand. Then, the extended omnibus model is given by

$$V_t(x) = \sum_{j\in\mathcal{J}} \sum_{d=0}^{\infty} p_{jdt} \max_{u_t\in\mathcal{U}(d)} \left\{ r_j^\top u_t + V_{t-1}(x + u_t) \right\} \quad \forall t \in \mathcal{T}, x \in \mathcal{X} \tag{3}$$

$$V_0(x) = \bar{r} \sum_{i\in\mathcal{I}} \min\{0, c_i - x_i\} \quad \forall x \in \mathcal{X}, \tag{4}$$

where $\bar{r}$ is the denied-boarding penalty. It is assumed greater than all fares, i.e., $\bar{r} > \max_{j,i}\{r_j^i\}$ so that the boundary conditions (4) ensure that overbooking any resource is never optimal.

2.3. The dynamic and the static two-leg revenue management model with flexible products

If a dynamic demand distribution (1) is given, the resulting specialization of the value function (3) which we refer to as dynamic model is given by

$$\text{(RMFP-d)} \qquad V_t(x) = \sum_{j\in\mathcal{J}} \sum_{d=0}^{1} p_{jdt} \max_{u_t\in\mathcal{U}(d)} \left\{ r_j^\top u_t + V_{t-1}(x + u_t) \right\} \quad \forall t \in \mathcal{T}, x \in \mathcal{X}.$$

Let $\mathbf{0}$ denote a zero vector and e_i denote a unit vector with value one at the ith position. Since no more than one request at a time is possible, the set of feasible actions in (RMFP-d) is

$$\mathcal{U}(0) = \{\mathbf{0}\} \quad \text{and} \quad \mathcal{U}(1) = \{\mathbf{0}, e_1, e_2\}.$$

Note the two major differences between our dynamic model and the one of Chen *et al.* (2010). First, (RMFP-d) does not require the fares of a flexible product to be equal to the fares of its constituting specific products. As a result, our model applies to instances with opaque products. Second, we do not need the formal distinction between specific and flexible product types at all because our model treats specific products technically as if they were flexible products. Our definition of the fare vectors r_j ensures that choosing to serve a request for product $j \in \mathcal{J}$ via $u_t = e_1$ if $r_j^1 = 0$ or via $u_t = e_2$ if $r_j^2 = 0$ is always inferior to rejecting it; otherwise, we would give away a seat for zero revenue. This modeling approach pays off later when we analyze the various monotonicity properties in Section 4.

For static demand distributions (2), the resulting specialization of (3) which we refer to as static model is given by

$$\text{(RMFP-s)} \qquad V_t(x) = \sum_{d=0}^{\infty} p_{jdt} \max_{u_t \in \mathcal{U}(d)} \left\{ r_j^\top u_t + V_{t-1}(x + u_t) \right\} \quad \forall t \in \mathcal{T}, j = j(t), x \in \mathcal{X}.$$

Here, the set of feasible actions can be written as

$$\mathcal{U}(d) = \{ve_1 + we_2 : v + w \leq d \text{ and } v, w \in \mathbb{Z}_+\} \quad \forall d \geq 0.$$

3. Event-based operators and reformulations

In this section, we introduce equivalent reformulations of the dynamic programs (RMFP-d) and (RMFP-s) that will be used in Section 4. These reformulations are obtained by defining control operators to handle single-request events and batch-request events.

We use the basic notation $P(f)$ for these event-based operators. It means that the operator P transforms the function f into a new function $P(f)$. $P(f)(x)$ means that the transformed function $P(f)$ is applied to some point x. In the following, all (transformed) functions have domain $\mathbb{Z}_+^2$.

3.1. Reformulation using single-request operators

Let $H^0(V) = V$ denote the identity operator. For the single-request event $d = 1$, we define the operator

$$H^1(V)(x) = \max \left\{ r^1 + V(x + e_1), r^2 + V(x + e_2), V(x) \right\} \quad \forall x \in \mathbb{Z}_+^2. \tag{5}$$

We define $r = (r^1, r^2)^\top$ and use the substitution $V = V_{t-1}, H^d = H_j^d$, and $r = r_j$ in (5) in order to plug the resulting equality into (RMFP-d). This yields the equivalent dynamic program

$$V_t(x) = \sum_{j \in \mathcal{J}} \sum_{d=0}^{1} p_{jdt} \, H_j^d(V_{t-1})(x) \quad \forall t \in \mathcal{T}, x \in \mathcal{X}. \tag{6}$$

The equivalence is obvious, since the operators $H_j^0(V_{t-1})(x)$ and $H_j^1(V_{t-1})(x)$ are by definition equal to the respective term on the right hand side of (RMFP-d) for all $j \in \mathcal{J}$, $t \in \mathcal{T}$, and $x \in \mathcal{X}$.

3.2. Reformulation using batch-request operators

For batch-request events, we generalize (5) by defining for $d \geq 1$ the operator

$$H^d(V)(x) = \max\left\{r^1 + H^{d-1}(V)(x+e_1), r^2 + H^{d-1}(V)(x+e_2), H^{d-1}(V)(x)\right\} \quad \forall x \in \mathbb{Z}_+^2. \tag{7}$$

The operator (7) recursively splits up the original batch request into multiple batches of size one. The partial fulfillment assumption allows us to state the next lemma.

Lemma 1. *Let any $x \in \mathbb{Z}_+^2$ be given. The following statement holds:*

$$H^d(V)(x) = \max_{u \in \mathcal{U}(d)} \left\{r^\top u + V(x+u)\right\} \quad \forall d \in \mathbb{Z}_+.$$

Proof. For demands $d \in \{0,1\}$, the result is trivial. For demands $d > 1$, we prove the result by induction over the demand levels. Assume that the result holds for $d-1$. Then, we have

$$\begin{aligned}
H^d(V)(x) &= \max\left\{r^1 + H^{d-1}(V)(x+e_1), r^2 + H^{d-1}(V)(x+e_2), H^{d-1}(V)(x)\right\} \\
&= \max\left\{\begin{array}{l} r^1 + \max\limits_{u \in \mathcal{U}(d-1)} \left\{r^\top u + V(x+u+e_1)\right\}, \\ r^2 + \max\limits_{u \in \mathcal{U}(d-1)} \left\{r^\top u + V(x+u+e_2)\right\}, \\ \qquad \max\limits_{u \in \mathcal{U}(d-1)} \left\{r^\top u + V(x+u)\right\} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} \max\limits_{u \in \mathcal{U}(d-1)} \left\{r^\top (u+e_1) + V(x+u+e_1)\right\}, \\ \max\limits_{u \in \mathcal{U}(d-1)} \left\{r^\top (u+e_2) + V(x+u+e_2)\right\}, \\ \max\limits_{u \in \mathcal{U}(d-1)} \left\{r^\top u + V(x+u)\right\} \end{array}\right\} \\
&= \max_{u \in \mathcal{U}(d)} \left\{r^\top u + V(x+u)\right\}.
\end{aligned}$$

The first equality is the definition (7), the second equality follows from the induction assumption, and the third equality follows from simple rearrangement. To see the last equality, we define for vectors v and for sets $\mathcal{M}$ of vectors that $\mathcal{M} + v = \{m + v : m \in \mathcal{M}\}$. Noting the equality

$$\mathcal{U}(d) = \big(\mathcal{U}(d-1) + e_1\big) \cup \big(\mathcal{U}(d-1) + e_2\big) \cup \mathcal{U}(d-1)$$

completes the proof. $\square$

In order to apply Lemma 1, we define (7) with $V = V_{t-1}, H^d = H^d_j$, and $r = r_j$. Plugging the resulting definition into (RMFP-s) gives the equivalent dynamic program

$$V_t(x) = \sum_{d=0}^{\infty} p_{jdt}\, H^d_j(V_{t-1})(x) \quad \forall t \in \mathcal{T}, j = j(t), x \in \mathcal{X}. \tag{8}$$

3.3. Universal lower and upper bounds

In what follows, we construct lower and upper bounds for the single-request operator (5). For $x \in \mathbb{Z}_+^2$, define the single-mode operators

$$T_1(f)(x) = \max\left\{r^1 + f(x+e_1), f(x)\right\} \text{ and } T_2(f)(x) = \max\left\{r^2 + f(x+e_2), f(x)\right\}.$$

Moreover, define the double-mode operator

$$H(f)(x) = \max\left\{T_1(f)(x), T_2(f)(x)\right\} \quad \forall x \in \mathbb{Z}_+^2. \tag{9}$$

Lemma 2. *Let $f : \mathbb{Z}_+^2 \mapsto \mathbb{R}$ be a real-valued function. The following inequalities hold for $x, y \in \mathbb{Z}_+^2$:*

$$\begin{aligned} &\min\left\{T_1(f)(x) - T_1(f)(y), T_2(f)(x) - T_2(f)(y)\right\} \\ &\qquad \le H(f)(x) - H(f)(y) \le \max\left\{T_1(f)(x) - T_1(f)(y), T_2(f)(x) - T_2(f)(y)\right\}. \end{aligned} \tag{10}$$

Proof. Choose any $x, y \in \mathbb{Z}_+^2$. Consider the case $H(f)(y) = T_1(f)(y)$. We have

$$H(f)(x) - H(f)(y) \ge T_1(f)(x) - T_1(f)(y).$$

because $H(f)(x) \ge T_1(f)(x)$. The case $H(f)(y) = T_2(f)(y)$ leads to

$$H(f)(x) - H(f)(y) \ge T_2(f)(x) - T_2(f)(y).$$

because $H(f)(x) \ge T_2(f)(x)$. Both inequalities above imply the first "$\le$" in (10). Similarly, the second "$\le$" in (10) follows from the analysis of the cases $H(f)(x) = T_1(f)(x)$ and $H(f)(x) = T_2(f)(x)$. The proof is complete as our choice of x, y was arbitrary. □

In the next lemma, we establish bounds for the single-mode operators $T_1(f)$ and $T_2(f)$.

Lemma 3. *Let $f : \mathbb{Z}_+^2 \mapsto \mathbb{R}$ be a real-valued function. For all $i \in \{1, 2\}$ and $x, y \in \mathbb{Z}_+^2$, we have*

$$\begin{aligned} &\min\{f(x) - f(y), f(x+e_i) - f(y+e_i)\} \\ &\qquad \le T_i(f)(x) - T_i(f)(y) \le \max\{f(x) - f(y), f(x+e_i) - f(y+e_i)\}. \end{aligned} \tag{11}$$

Proof. Similar to the proof of Lemma 2. □

Lemmata 2 and 3 will be helpful to analyze the time monotonicity in the following section.

4. Monotonicity properties

The objective of this section is to show that the value functions defined by the models (RMFP-d) and (RMFP-s) are monotonic with respect to changes in inventory and time. First, the properties concavity, submodularity, subconcavity, and multimodularity are defined, and some equivalent conditions are presented. Our definitions are in line with those presented in Zhuang and Li (2010).

4.1. Definitions and equivalent conditions

Let $\Delta_i f(x) = f(x) - f(x+e_i)$ for $i \in \{1,2\}$ denote the *forward difference operator.*

Definition 1. *Let $f : \mathbb{Z}_+^2 \mapsto \mathbb{R}$ be a real-valued function. The function f is*

(i) *(discrete) concave in x_i if $\Delta_i f(x) \leq \Delta_i f(x+e_i)$ for all $x \in \mathbb{Z}_+^2$ and $i \in \{1,2\}$,*

(ii) *submodular in x_i and x_h if $\Delta_i f(x) \leq \Delta_i f(x+e_h)$ for all $x \in \mathbb{Z}_+^2$ and $i,h \in \{1,2\}, i \neq h$,*

(iii) *subconcave in x_i and x_h if $\Delta_i \Delta_i f(x) \leq \Delta_i \Delta_h f(x)$ for all $x \in \mathbb{Z}_+^2$ and $i,h \in \{1,2\}, i \neq h$.*

The properties of Definition 1 are global. For example, if a function $f(x)$ is concave in x_i, this statement holds for the entire domain $\mathbb{Z}_+^2$ of $f(x)$. If we just say that f is concave, this means that $f(x)$ is component-wise concave. A submodular/subconcave function f means that $f(x)$ is submodular/subconcave in all distinct pairs of components of x.

Observe that the linear operator Δ_i satisfies

$$\Delta_i \Delta_h f(x) = \Delta_h \Delta_i f(x) \quad \forall i,h \in \{1,2\}, i \neq h.$$

Using the above equality and the definition of submodularity, the equivalent conditions for submodular functions follow directly and are stated without proof.

Lemma 4. *The following statements are equivalent:*

1. *The function $f : \mathbb{Z}_+^2 \mapsto \mathbb{R}$ is submodular.*
2. *$\Delta_i \Delta_h f(x) \leq 0$ for all $x \in \mathbb{Z}_+^2$ and $i,h \in \{1,2\}, i < h$.*
3. *$f(x) - f(x+e_i)$ nondecreasing in x_h for all $x \in \mathbb{Z}_+^2$ and $i,h \in \{1,2\}, i < h$.*

Lemma 4 states that a function which is submodular in x_i and x_h is automatically submodular in x_h and x_i (vice versa). The following equivalent conditions hold for subconcave functions.

Lemma 5. *The following statements are equivalent:*

1. *The function $f : \mathbb{Z}_+^2 \mapsto \mathbb{R}$ is subconcave.*
2. *$\Delta_1 f(x+e_2) - \Delta_2 f(x+e_2) \leq \Delta_1 f(x) - \Delta_2 f(x) \leq \Delta_1 f(x+e_1) - \Delta_2 f(x+e_1)$ for all $x \in \mathbb{Z}_+^2$.*
3. *$f(x+e_i) - f(x+e_h)$ nondecreasing in x_h for all $x \in \mathbb{Z}_+^2$ and $i,h \in \{1,2\}, i \neq h$.*
4. *$f(x+e_i) - f(x+e_h)$ nonincreasing in x_i for all $x \in \mathbb{Z}_+^2$ and $i,h \in \{1,2\}, i \neq h$.*

Proof. Given any $i,h \in \{1,2\}, i \neq h$, it follows that for any $x \in \mathbb{Z}_+^2$

$$\begin{aligned} & \Delta_i \Delta_i f(x) \leq \Delta_i \Delta_h f(x) \\ \Leftrightarrow \quad & \Delta_i f(x) - \Delta_i f(x+e_i) \leq \Delta_h f(x) - \Delta_h f(x+e_i) \\ \Leftrightarrow \quad & \Delta_i f(x) - \Delta_h f(x) \leq \Delta_i f(x+e_i) - \Delta_h f(x+e_i) \\ \Leftrightarrow \quad & f(x+e_h) - f(x+e_i) \leq f(x+e_i+e_h) - f(x+2e_i). \end{aligned}$$

The last inequality is part 3. and multiplying by -1 yields part 4. We obtain part 2. by taking $i = 1, h = 2$ and $h = 1, i = 2$ in the third inequality above. This completes the proof. ☐

The linearity of the forward difference operator and the conditions for concavity, submodularity, and subconcavity have the following consequence.

Lemma 6. *For real-valued functions $f_k : \mathbb{Z}_+^2 \mapsto \mathbb{R}, k = 1, \dots, \kappa$ and nonnegative weights $\alpha_k, k = 1, \dots, \kappa$ the following statements hold:*

1. *If all f_k are concave, so is $\sum_{k=1}^{\kappa} \alpha_k f_k$.*
2. *If all f_k are submodular, so is $\sum_{k=1}^{\kappa} \alpha_k f_k$.*
3. *If all f_k are subconcave, so is $\sum_{k=1}^{\kappa} \alpha_k f_k$.*

We are now able to define multimodular functions.

Definition 2. *Let $f : \mathbb{Z}_+^2 \mapsto \mathbb{R}$ be a real-valued function. The function f is called multimodular if it is submodular and subconcave.*

The reason why we consider multimodular functions is the fact that submodularity and subconcavity together imply concavity (see Definition 1). For this reason, a proof of concavity is redundant. Zhuang and Li (2012) provide a broader view on multimodular functions. Morton (2006) considers directional modularity which is a similar concept.

4.2. Monotonicity properties of the dynamic model

To examine the multimodularity of the value function defined by (RMFP-d), we make use of the equivalent dynamic programming formulation (6). Therefore, we start with the following analysis of the single-request operator (5).

Lemma 7. *Let any $j \in \mathcal{J}$ and $t \in \mathcal{T}$ be given, and let V_{t-1} be defined by (RMFP-d). If V_{t-1} is multimodular, then $H_j^1(V_{t-1})$ is multimodular.*

Proof. By Definition 2, this lemma builds on the fact that $H_j^1(V_{t-1})$ preserves the (i) submodularity and (ii) subconcavity which are both induced by V_{t-1}. The proof of (i) and (ii) is a longer case-by-case analysis and therefore presented in Appendix A. □

The above lemma states that the single-request operator (5) preserves multimodularity within a given period. We next show that multimodulartiy is propagated over time.

Lemma 8. *Let $V_t, t \in \mathcal{T}$ be defined by (RMFP-d). V_t is multimodular for all $t \in \mathcal{T}$.*

Proof. The proof is by induction over the time periods. For $t = 0$, the boundary conditions (4) imply the submodularity

$$V_0(x) - V_0(x + e_i) - V_0(x + e_h) + V_0(x + e_i + e_h) = 0$$

and the subconcavity

$$V_0(x + e_i) - V_0(x + e_h) - V_0(x + e_i + e_h) + V_0(x + 2e_h) = \begin{cases} -\bar{r} & \text{if } x_h = c_h - 1 \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in \mathcal{X}$ and $i, h \in \{1, 2\}, i \neq h$. This shows that V_0 is multimodular, see Definition 2.

Assume that the result holds for $t - 1$, i.e., V_{t-1} is multimodular and herewith $H_j^0(V_{t-1})$. Using the value function (6), Lemma 6, and Lemma 7 yields that V_t is multimodular. Repeating this argument for all $t \in \mathcal{T}$ completes the proof. □

The following lemma states that the value function is submodular with respect to changes in time meaning that the marginal seat revenues decrease as time approaches the end of the booking horizon. We use the standard shorthand notation $[x]^+ = \max\{x, 0\}$.

Lemma 9. *Let $V_t, t \in \mathcal{T}$ be defined by (RMFP-d). The following statement holds: $\Delta_i V_t(x)$ is nondecreasing in t for all $x \in \mathcal{X}$ and $i \in \mathcal{I}$.*

Proof. The condition of this lemma can be equivalently stated as:

$$\Delta_i V_t(x) - \Delta_i V_{t-1}(x) \geq 0 \quad \forall i \in \mathcal{I}, t \in \mathcal{T}, x \in \mathcal{X}. \tag{12}$$

Let $F_1(V)(x) = \left[r^1 - \Delta_1 V(x)\right]^+$ and $F_2(V)(x) = \left[r^2 - \Delta_2 V(x)\right]^+$ for all $x \in \mathbb{Z}_+^2$. Moreover, we define the identity operator $G^0(V_{t-1}) = \Delta V_{t-1}$ and the operator

$$\begin{aligned} G^1(V)(x) = & \max\left\{r^1 - \Delta_1 V(x), r^2 - \Delta_2 V(x), 0\right\} \\ = & \max\left\{F_1(V)(x), F_2(V)(x)\right\} \quad \forall x \in \mathbb{Z}_+^2. \end{aligned} \tag{13}$$

It is straightforward to show that the right hand side of (RMFP-d) can be expressed as

$$V_t(x) = V_{t-1}(x) + \sum_{j \in \mathcal{J}} p_{j1t} \max\{r_j^1 - \Delta_1 V_{t-1}(x), r_j^2 - \Delta_2 V_{t-1}(x), 0\} \tag{14}$$

$$= V_{t-1}(x) + \sum_{j \in \mathcal{J}} p_{j1t}\, G_j^1(V_{t-1})(x) \quad \forall t \in \mathcal{T}, x \in \mathcal{X}, \tag{15}$$

where the second equality follows by setting $G^1 = G_j^1, V = V_{t-1}$, and $r = r_j$. Using the definition of Δ_i and the equality (15), the left hand side of (12) can be rewritten as

$$\Delta_i V_t(x) - \Delta_i V_{t-1}(x) = \sum_{j \in \mathcal{J}} p_{j1t} \Big[G_j^1(V_{t-1})(x) - G_j^1(V_{t-1})(x + e_i) \Big] \quad \forall i \in \mathcal{I}, t \in \mathcal{T}, x \in \mathcal{X}.$$

It remains to show that the difference within brackets is nonnegative. Setting $G^1 = G_j^1, V = V_{t-1}, r = r_j$, and $y = x + e_i$, it follows that

$$\begin{aligned} & G^1(V)(x) - G^1(V)(x + e_i) \\ & \quad \geq \min\Big\{ F_1(V)(x) - F_1(V)(y), F_2(V)(x) - F_2(V)(y) \Big\} \\ & \quad = \min\Big\{ T_1(V)(x) - T_1(V)(y), T_2(V)(x) - T_2(V)(y) \Big\} - V(x) + V(y) \\ & \quad \geq \min\Big\{ V(x) - V(y), V(x + e_1) - V(y + e_1), V(x + e_2) - V(y + e_2) \Big\} - V(x) + V(y) \\ & \quad = \min\Big\{ 0, \Delta_1 V(y) - \Delta_1 V(x), \Delta_2 V(y) - \Delta_2 V(x) \Big\} \\ & \quad = \min\Big\{ 0, \Delta_1 V(x + e_i) - \Delta_1 V(x), \Delta_2 V(x + e_i) - \Delta_2 V(x) \Big\} \\ & \quad \geq 0 \qquad \forall i \in \{1, 2\}, x \in \mathbb{Z}_+^2, \end{aligned}$$

where the first inequality follows from the first inequality of Lemma 2, the first equality results from $F_i(V)(x) = T_i(V)(x) - V(x)$, the second inequality is a consequence of the first inequality of Lemma 3, the second equality holds by simple rearrangement. The last equality is obvious and the last inequality can be derived whenever V is concave and submodular, see Definition 1. Since Lemma 8 has shown that V_t is multimodular for all $t \in \mathcal{T}$, the proof is complete. □

We stress that the time monotonicity relies only on the concavity and submodularity of the value function. This result does not require the value function to be subconcave.

4.3. Monotonicity properties of the static model

In the following, the multimodularity of the value function defined by (RMFP-s) is analyzed.

Lemma 10. *Let any $j \in \mathcal{J}$ and $t \in \mathcal{T}$ be given, and let V_{t-1} be defined by (RMFP-s). If V_{t-1} is multimodular, then $H_j^d(V_{t-1})$ is multimodular for all $d \in \mathbb{Z}_+$.*

Proof. We show the result by induction over the demand levels. $H^0(V_{t-1}) = V_{t-1}$ is multimodular by assumption. Consider $d \geq 1$ and assume that the result holds for $d-1$. Using $H^d(V) = H^1(H^{d-1}(V))$ and that H^1 preserves multimodularity (see Lemma 7) completes the proof. □

Lemma 11. *Let $V_t, t \in \mathcal{T}$ be defined by (RMFP-s). V_t is multimodular for all $t \in \mathcal{T}$.*

Proof. Applying Lemma 1, the proof is the same as that of Lemma 8 except that here we apply Lemma 10 to the equivalent value function (8). □

Note that Lemma 11 applies to any arrival sequence that satisfies (2), since we did not specify a particular ordering of products. Finally, we analyze the time monotonicity.

Lemma 12. *Let $V_t, t \in \mathcal{T}$ be defined by (RMFP-s). The following statement holds: $\Delta_i V_t(x)$ is nondecreasing in t for all $x \in \mathcal{X}$ and $i \in \mathcal{I}$.*

Proof. We need to verify that $\Delta_i V_t(x) - \Delta_i V_{t-1}(x) \geq 0$ for all $i \in \mathcal{I}, t \in \mathcal{T}$, and $x \in \mathcal{X}$. Using the notation defined in the proof of Lemma 9, we generalize (13) by defining the operator

$$\begin{aligned} G^d(V)(x) &= \max\left\{r^1 - \Delta_1 H^{d-1}(V)(x), r^2 - \Delta_2 H^{d-1}(V)(x), 0\right\} \\ &= \max\{F_1(H^{d-1}(V)), F_2(H^{d-1}(V))\} \qquad \forall x \in \mathbb{Z}_+^2, d \geq 1. \end{aligned}$$

It is easy to show that the right hand side (RHS) of the equivalent value function (8) can be expressed as

$$\begin{aligned} V_t(x) &= V_{t-1}(x) + \sum_{d=1}^{\infty} p_{jdt} \Bigg[\Big[H_j^{d-1}(V_{t-1})(x) - H_j^0(V_{t-1})(x) \Big] \\ &\qquad + \max\left\{ r_j^1 - \Delta_1 H_j^{d-1}(V_{t-1})(x), r_j^2 - \Delta_2 H_j^{d-1}(V_{t-1})(x), 0 \right\} \Bigg] \\ &= V_{t-1}(x) + \sum_{d=1}^{\infty} p_{jdt} \Bigg[\Big[H_j^{d-1}(V_{t-1})(x) - H_j^0(V_{t-1})(x) \Big] + G_j^d(V_{t-1})(x) \Bigg] \\ &= V_{t-1}(x) + \sum_{d=1}^{\infty} p_{jdt} \Bigg[\sum_{z=1}^{d-1} \Big[H_j^z(V_{t-1})(x) - H_j^{z-1}(V_{t-1})(x) \Big] + G_j^d(V_{t-1})(x) \Bigg] \\ &\qquad \forall t \in \mathcal{T}, j = j(t), x \in \mathcal{X}, \qquad (16) \end{aligned}$$

where the second equality follows from setting $G^d = G_j^d, V = V_{t-1}$, and $r = r_j$, and the third equality follows from formulating the difference (inner brackets) as a telescoping series. Now, we use equation (16) and the definition of Δ_i in order to rewrite $\Delta_i V_t(x) - \Delta_i V_{t-1}(x)$ as

$$\begin{aligned}
&V_t(x) - V_t(x+e_i) - \Delta_i V_{t-1}(x) \\
&\quad = \sum_{d=1}^{\infty} p_{kjdt} \left[\sum_{z=1}^{d-1} \Big[\Delta_i H_j^z(V_{t-1})(x) - \Delta_i H_j^{z-1}(V_{t-1})(x) \Big] \right. \\
&\qquad \left. + \Big[G_j^d(V_{t-1})(x) - G_j^d(V_{t-1})(x+e_i) \Big] \right] \quad \forall i \in \mathcal{I}, t \in \mathcal{T}, j = j(t), x \in \mathcal{X}. \qquad (17)
\end{aligned}$$

To analyze the nonnegativity of the RHS of (17), we first rewrite the difference $\Delta_i H_j^d(V_{t-1})(x) - \Delta_i H_j^{d-1}(V_{t-1})(x)$ using the definition of Δ_i as

$$\begin{aligned}
&H_j^d(V_{t-1})(x) - H_j^d(V_{t-1})(x+e_i) - \Big[H_j^{d-1}(V_{t-1})(x) - H_j^{d-1}(V_{t-1})(x+e_i) \Big] \\
&\quad = H_j^d(V_{t-1})(x) - H_j^{d-1}(V_{t-1})(x) - \Big[H_j^d(V_{t-1})(x+e_i) - H_j^{d-1}(V_{t-1})(x+e_i) \Big] \\
&\quad = \max \Big\{ r_j^1 - \Delta_1 H_j^{d-1}(V_{t-1})(x), r_j^2 - \Delta_2 H_j^{d-1}(V_{t-1})(x), 0 \Big\} \\
&\qquad - \max \Big\{ r_j^1 - \Delta_1 H_j^{d-1}(V_{t-1})(x+e_i), r_j^2 - \Delta_2 H_j^{d-1}(V_{t-1})(x+e_i), 0 \Big\} \\
&\quad = G_j^d(V_{t-1})(x) - G_j^d(V_{t-1})(x+e_i) \qquad \forall i \in \mathcal{I}, t \in \mathcal{T}, j = j(t), x \in \mathcal{X}, d \geq 1, \qquad (18)
\end{aligned}$$

where the second equality is obvious, the third equality follows from the definition of H^d, and the last inequality follows from the definition of G^d with $G^d = G_j^d, V = V_{t-1}$, and $r = r_j$. Consequently, if we verify the nonnegativity of the RHS of (18), this implies the nonnegativity of the RHS of (17) and completes the proof. With the equality $G_j^d(V_{t-1})(x) = G_j^1(H^{d-1}(V_{t-1}))(x)$, the RHS of (17) reads as

$$G_j^1(H^{d-1}(V_{t-1}))(x) - G_j^1(H^{d-1}(V_{t-1}))(x+e_i) \geq 0 \quad \forall i \in \mathcal{I}, t \in \mathcal{T}, j = j(t), x \in \mathbb{Z}_+^2, d \geq 1,$$

where the inequality can be derived similar to the proof of Lemma 9 using that H_j^d preserves multimodularity (see Lemmata 10 and 11). $\square$

5. Optimal booking policies

In this section, we derive optimal booking policies for both the dynamic and the static model of Section 2. To achieve this, we first introduce the notion of a booking path P by defining the sequence

$$P = \Big(e_{i(1)}, \ldots, e_{i(l)}, \ldots, e_{i(L)} \Big)$$

of unit vectors $e_{i(l)}$, where $i(l) \in \{1,2\} = \mathcal{I}$ denotes the lth path index and $L \in \mathbb{Z}_+$ denotes the length of that path. The different states visited by traversing P are defined as

$$x_{(l)} = x + \sum_{l'=1}^{l} e_{i(l')} \quad \forall l \in \{0, \ldots, L\}$$

so that $x_0 = x$ is the starting point and $x_{(L)}$ the endpoint of P. Note that $x_{(l)} = x_{(l-1)} + e_{i(l)}$ for $1 \leq l < L$.

In the remainder of this section, we show how to construct optimal booking paths, i.e., paths that characterize optimal booking decisions.

5.1. Optimal booking paths

We first describe the construction of booking paths for a given state $x \in \mathcal{X}$, product $j \in \mathcal{J}$, and time $t \in \mathcal{T}$. In the following, we often omit the indices j and t in order to lighten the notation. Let $r^i(x) = r^i_j - \Delta_i V_{t-1}(x)$ denote the marginal revenue increase obtained by allocating the x_ith seat of flight $i \in \mathcal{I}$. We base our decisions on two simple criteria: $\max\{r^1(x), r^2(x)\} > 0$ for the acceptance and $r^1(x) \geq r^2(x)$ for the assignment. Ties are broken by rejecting and by assigning to flight 1, respectively. Constructing a path amounts to choosing the path indices $i(l)$ for $l \geq 1$ by applying the above criteria. We choose $i(l) = \arg\max_{i\in\mathcal{I}} r^i(x_{(l-1)})$ as long as $\max\{r^1(x_{(l-1)}), r^2(x_{(l-1)})\} > 0$ and we stop the construction otherwise, see Algorithm 1.

A path $P = P_{jt}$ is valid for a period $t \in \mathcal{T}$ and product $j \in \mathcal{J}$ by construction but does not depend on the demand that we actually observe in t. This allows us to derive booking decisions for every possible demand level $d \in \mathbb{Z}_+$ for product j in t from the same path P_{jt}.

Algorithm 1: Path Construction

Input: $t \in \mathcal{T}, x \in \mathcal{X}, j \in \mathcal{J}, V_{t-1}$

$l := 1, x_0 := x$, and $P_{jt} := ()$
while $\max\{r^1(x_{(l-1)}), r^2(x_{(l-1)})\} > 0$ **do**
  if $r^1(x_{(l-1)}) \geq r^2(x_{(l-1)})$ **then** $i(l) = 1$ **else** $i(l) = 2$
  $P_{jt} := (P_{jt}, e_{i(l)})$
  $x_{(l)} := x_{(l-1)} + e_{i(l)}$
  $l := l + 1$
$L := l - 1$

Output: Path $P = P_{jt}$ of length L

For a path P and demand $d \in \mathbb{Z}_+$, we define booking decisions $u_t^{(l)} \in \mathcal{U}(d)$ as

$$u_t^{(l)} = \sum_{l'=1}^{\min\{d,l\}} e_{i(l')} \quad \forall l \in \{0, 1, \ldots, L\}, \tag{19}$$

in particular $u_t^{(0)} = \mathbf{0}$. Then, following path P means that $u_{1t}^{(l)}$ requests are assigned to flight 1 and $u_{2t}^{(l)}$ requests to flight 2 so that at most d requests are allocated.

We are now ready to state the main theorem of this section. The notation $R(u) = r_j^\top u + V_{t-1}(x + u) - V_{t-1}(x)$ simplifies the exposition.

Theorem 1. *Let any state $x \in \mathcal{X}$ at time $t \in \mathcal{T}$ and any demand $d \in \mathbb{Z}_+$ for product $j \in \mathcal{J}$ be given, and let P_{jt} be a booking path according to Algorithm 1. Define $\ell = \min\{d, L\}$. The booking decision $u_t^{(\ell)}$ is optimal in both the dynamic model (RMFP-d) and the static model (RMFP-s).*

Proof. Proof by contradiction, i.e., let $u_t^* \neq u_t^{(\ell)}$ be an optimal booking decision with $R(u_t^*) > R(u_t^{(\ell)})$. In particular, $R(u_t^*) \geq R(u)$ for all $u \in \mathcal{U}(d)$. Without loss of generality, we also assume that u_t^* is chosen with smallest possible norm $||u_t^*||_1$. Let $x^P = x + u_t^{(\ell)}$ denote the state that results from following P_{jt} up to the ℓth state and let $x^* = x + u_t^*$ denote the state resulting from u_t^*. In the following, we analyze eight possible cases, for clarity depicted in Table 2:

Case		$u_{1t}^{(\ell)}$ vs. u_{1t}^*		
		$>$	$=$	$<$
$u_{2t}^{(\ell)}$ vs. u_{2t}^*	$>$	(VIII)	(II)	(VI)
	$=$	(I)		(III)
	$<$	(V)	(IV)	(VII)

Table 2: Cases in the proof of Theorem 1

Case (I). $u_{1t}^{(\ell)} > u_{1t}^*$ and $u_{2t}^{(\ell)} = u_{2t}^*$. Suppose that $x_{(l)} = x^*$ for some $l \in \{0, \ldots, L-1\}$. From the construction of P_{jt} follows that $r^1(x^*)$ is positive and, hence,

$$R(u_t^* + e_1) = r^\top(u_t^* + e_1) + V_{t-1}(x + u_t^* + e_1) - V_{t-1}(x) = R(u_t^*) + r^1(x + u_t^*) > R(u_t^*),$$

which contradicts the optimality of u_t^*.

If $x^* \neq x_{(l)}$ for all $l \in \{0, \ldots, L-1\}$, consider the unique state $x_{(l)}$ with $x_{1l} = x_1^*$ having the largest l. Then, $e_{i(l+1)} = e_1$ by the definition of l and P_{jt}, i.e., $r^1(x_{(l)})$ is positive and greater or equal than $r^2(x_{(l)})$. We define $K = x_2^* - x_{2l}$ and use subconcavity to state that

$$\begin{aligned}
0 < r^1(x_{(l)}) - r^2(x_{(l)}) &= r_j^1 - r_j^2 + \Delta_2 V_{t-1}(x_{(l)}) - \Delta_1 V_{t-1}(x_{(l)}) \\
&\leq r_j^1 - r_j^2 + \Delta_2 V_{t-1}(x_{(l)} + e_2) - \Delta_1 V_{t-1}(x_{(l)} + e_2) \\
&\;\;\vdots \\
&\leq r_j^1 - r_j^2 + \Delta_2 V_{t-1}(x_{(l)} + (K-1)e_2) - \Delta_1 V_{t-1}(x_{(l)} + (K-1)e_2) \\
&= r^1(x^* - e_2) - r^2(x^* - e_2).
\end{aligned} \tag{20}$$

Hence, $R(u_t^* + e_1 - e_2) = R(u_t^*) + r^1(x^* - e_2) - r^2(x^* - e_2) > R(u_t^*)$ is true and contradicts the optimality of u_t^*.

Case (II). $u_{1t}^{(\ell)} = u_{1t}^*$ and $u_{2t}^{(\ell)} > u_{2t}^*$. The contradiction is obtained by interchanging the roles of $i = 1$ and $i = 2$.

Case (III). $u_{1t}^{(\ell)} < u_{1t}^*$ and $u_{2t}^{(\ell)} = u_{2t}^*$. If $r^1(x^P) > 0$, then $u_{1t}^{(\ell)} + u_{2t}^{(\ell)} = d$ follows from the definition of $u_t^{(\ell)}$. Since $u_{1t}^* + u_{2t}^* > u_{1t}^{(\ell)} + u_{2t}^{(\ell)}$, u_t^* is infeasible ($u_t^* \notin \mathcal{U}(d)$) and herewith not optimal.

Suppose that $r^1(x^P) \leq 0$ and set $K = x_1^* - x_1^P$. Concavity of V implies that

$$0 \geq r_j^1 - \Delta_1 V_{t-1}(x^P) \geq r_j^1 - \Delta_1 V_{t-1}(x^P + e_1) \geq \cdots \geq r_j^1 - \Delta_1 V_{t-1}(x^P + (K-1)e_1) = r^1(x^* - e_1).$$

The relation $R(u_t^* - e_1) = R(u_t^*) - r^1(x^* - e_1) \geq R(u_t^*)$ means that $u_t^* - e_1$ is another optimal solution which, however, contradicts the smallest norm assumption of u_t^*.

Case (IV). $u_{1t}^{(\ell)} = u_{1t}^*$ and $u_{2t}^{(\ell)} < u_{2t}^*$. The contradiction is obtained by interchanging the roles of $i = 1$ and $i = 2$.

Case (V). $u_{1t}^{(\ell)} > u_{1t}^*$ and $u_{2t}^{(\ell)} < u_{2t}^*$. Consider the unique state $x_{(l)}$ with $x_{1l} = x_1^*$ and largest l. Then, $e_{i(l+1)} = e_1$ by the definition of l and P_{jt}. Hence, $r^1(x_{(l)}) > r^2(x_{(l)})$ and the result in (20) with $K = x_2^* - x_{2l}$ yields the contradiction $R(u_t^* + e_1 - e_2) > R(u_t^*)$.

Case (VI). $u_{1t}^{(\ell)} < u_{1t}^*$ and $u_{2t}^{(\ell)} > u_{2t}^*$. Again, the contradiction is obtained by interchanging the roles of $i = 1$ and $i = 2$.

Case (VII). $u_{1t}^{(\ell)} < u_{1t}^*$ and $u_{2t}^{(\ell)} < u_{2t}^*$. As in Cases (III) and (IV), $r^1(x^P) > 0$ or $r^2(x^P) > 0$ implies $u_{1t}^{(\ell)} + u_{2t}^{(\ell)} = d$, $u_{1t}^* + u_{2t}^* > u_{1t}^{(\ell)} + u_{2t}^{(\ell)}$, u_t^* is infeasible ($u_t^* \notin \mathcal{U}(d)$) and herewith not optimal.

Therefore, $r^1(x^P) \leq 0$ and $r^2(x^P) \leq 0$. Set $K_1 = x_1^* - x_1^P$ and $K_2 = x_2^* - x_2^P$. Submodularity of V implies that

$$0 \geq r_j^1 - \Delta_1 V_{t-1}(x^P) \geq r_j^1 - \Delta_1 V_{t-1}(x^P + e_2) \geq \cdots \geq r_j^1 - \Delta_1 V_{t-1}(x^P + K_2 e_2)$$

and concavity of V implies that

$$\begin{aligned} 0 &\geq r_j^1 - \Delta_1 V_{t-1}(x^P + K_2 e_2) \geq r_j^1 - \Delta_1 V_{t-1}(x^P + e_1 + K_2 e_2) \\ &\geq \cdots \geq r_j^1 - \Delta_1 V_{t-1}(x^P + (K_1 - 1)e_1 + K_2 e_2) = r^1(x^* - e_1). \end{aligned}$$

The relation $R(u_t^* - e_1) = R(u_t^*) - r^1(x^* - e_1) \geq R(u_t^*)$ means that $u_t^* - e_1$ is another optimal solution which, however, contradicts the smallest norm assumption of u_t^*.

Case (VIII). $u_{1t}^{(\ell)} > u_{1t}^*$ and $u_{2t}^{(\ell)} > u_{2t}^*$. If $x_{(l)} = x^*$ for some $l \in \{0, \ldots, L-1\}$, then by construction of P_{jt} we know that $r^1(x^*)$ or $r^2(x^*)$ is positive and, hence, either

$$R(u_t^* + e_1) = r^\top(u_t^* + e_1) + V_{t-1}(x + u_t^* + e_1) - V_{t-1}(x) = R(u_t^*) + r^1(x^*) > R(u_t^*),$$

or

$$R(u_t^* + e_2) = r^\top(u_t^* + e_2) + V_{t-1}(x + u_t^* + e_2) - V_{t-1}(x) = R(u_t^*) + r^2(x^*) > R(u_t^*),$$

which contradicts the optimality of u_t^*.

Otherwise, consider the unique state $x_{(l)}$ with $x_{1l} \leq x_1^*$ and $x_{2l} \leq x_2^*$ having the largest l. If $e_{i(l+1)} = e_1$, the definition of l and P_{jt} implies that $r^1(x_{(l)})$ is positive and greater or equal than $r^2(x_{(l)})$. As in Case (I), the inequality (20) then yields $R(u_t^* + e_1 - e_2) > R(u_t^*)$, which contradicts the optimality of u_t^*. The alternative $e_{i(l+1)} = e_2$ leads to the symmetric contradiction as covered by Case (II). This completes the proof. □

5.2. *Relationship with Chen* et al.*'s switching curves*

Chen *et al.* (2010) describe their optimal booking policy based on two types of information. First, acceptance or rejection requires the evaluation of one type of switching curve which is a function of a single state variable. Second, in case of acceptance, the assignment requires the evaluation of another type of switching curve which is again a function of a single state variable.

Based on Theorem 1, we can describe those states $x \in \mathcal{X}$ in which it is optimal to accept at least one unit of demand for product $j \in \mathcal{J}$ at time $t \in \mathcal{T}$ as the following acceptance region

$$\mathcal{X}_{jt}^{acc} = \big\{x \in \mathcal{X} : \max\{r_j^1 - \Delta_1 V_{t-1}(x), r_j^2 - \Delta_2 V_{t-1}(x)\} > 0\big\}.$$

In the static model (RMFP-s), the only non-empty acceptance region is the one for $j = j(t)$.

Our definition of *switching curves* for acceptance and a given value function V is as follows:

$$\begin{aligned} S_1(x_2) &= \min\Big\{\hat{x}_1 \in \mathcal{X}_1 : \max\big\{r^1 - \Delta_1 V(\hat{x}_1, x_2), r^2 - \Delta_2 V(\hat{x}_1, x_2)\big\} \leq 0\Big\} \quad \forall x_2 \in \mathcal{X}_2 \\ S_2(x_1) &= \min\Big\{\hat{x}_2 \in \mathcal{X}_2 : \max\big\{r^1 - \Delta_1 V(\hat{x}_2, x_1), r^2 - \Delta_2 V(\hat{x}_2, x_1)\big\} \leq 0\Big\} \quad \forall x_1 \in \mathcal{X}_1, \end{aligned}$$

where $\mathcal{X}_1 = \mathcal{X}_2 = \mathbb{Z}_+$.

Lemma 13. *Let V be concave and submodular and $x \in \mathbb{Z}_+^2$. Then,*

(i) $x_1 \geq S_1(x_2)$ *implies* $\max\{r^1 - \Delta_1 V(x_1, x_2), r^2 - \Delta_2 V(x_1, x_2)\} \leq 0$,

(ii) $x_2 \geq S_2(x_1)$ *implies* $\max\{r^1 - \Delta_1 V(x_1, x_2), r^2 - \Delta_2 V(x_1, x_2)\} \leq 0$,

(iii) $x_1 < S_1(x_2)$ *if and only if* $x_2 < S_2(x_1)$.

Proof. (i) By definition of S_1, we have $r^1 - \Delta_1 V(S_1(x_2), x_2) \leq 0$. Concavity of V implies $r^1 - \Delta_1 V(x_1, x_2) \leq 0$ for all $x_1 \geq S_1(x_2)$. By definition of S_1, we also know that $r^2 - \Delta_2 V(S_1(x_2), x_2) \leq 0$. Submodularity of V implies $r^2 - \Delta_2 V(x_1, x_2) \leq 0$ for all $x_1 \geq S_1(x_2)$. Both inequalities together imply $\max\{r^1 - \Delta_1 V(x_1, x_2), r^2 - \Delta_2 V(x_1, x_2)\} \leq 0$, i.e., the statement.
(ii) Follows from (i) with the roles of $i = 1$ and $i = 2$ interchanged.
(iii) Now, the condition $x_1 < S_1(x_2)$ is equivalent to the condition $\Delta_1 V(x_1, x_2) - r^1 > 0$ *or* $\Delta_2 V(x_1, x_2) - r^2 > 0$ which is again equivalent to $x_2 < S_2(x_1)$ by the definitions of S_1 and S_2. ☐

A direct consequence of Lemma 13 and the time monotonicity proven in Lemmata 9 and 12 is:

Theorem 2. *Let the switching curves $S_1 = S_{1jt}$ and $S_2 = S_{2jt}$ be defined with $V = V_{t-1}$ and $r = r_j$ for products $j \in \mathcal{J}$ and periods $t \in \mathcal{T}$. Then, the acceptance regions in the models (RMFP-s) and (RMFP-d) are*

$$\mathcal{X}_{jt}^{acc} = \left\{(x_1, x_2)^\top \in \mathcal{X} : x_1 < S_{1jt}(x_2)\right\} = \left\{(x_1, x_2)^\top \in \mathcal{X} : x_2 < S_{2jt}(x_1)\right\}$$

for all products $j \in \mathcal{J}$ and periods $t \in \mathcal{T}$. Moreover, $\mathcal{X}_{jt}^{acc} \subseteq \mathcal{X}_{jt-1}^{acc}$ for all $t \in \mathcal{T}$.

Another implication of Theorem 1 is that we can describe those states $x \in \mathcal{X}$ in which it is better to assign one unit of demand for product $j \in \mathcal{J}$ at time $t \in \mathcal{T}$ to the respective flight. Assignments to flight 1 are more attractive in states given by the assignment region

$$\mathcal{X}_{1jt}^{ass} = \left\{x \in \mathcal{X} : r_j^1 - \Delta_1 V_{t-1}(x) \geq r_j^2 - \Delta_2 V_{t-1}(x)\right\} \quad \forall j \in \mathcal{J}, t \in \mathcal{T},$$

while assignments to flight 2 are more attractive in the following states

$$\mathcal{X}_{2jt}^{ass} = \left\{x \in \mathcal{X} : r_j^1 - \Delta_1 V_{t-1}(x) < r_j^2 - \Delta_2 V_{t-1}(x)\right\} \quad \forall j \in \mathcal{J}, t \in \mathcal{T}.$$

Note that here assignment is made independent of acceptance. In the booking policy, assignments are made only if demand has been accepted.

We define associated switching curves for assignments provided that a value function V is given:

$$Q_1(x_2) = \min\left\{\hat{x}_1 \in \mathcal{X}_1 : r^1 - \Delta_1 V(\hat{x}_1, x_2) < r^2 - \Delta_2 V(\hat{x}_1, x_2)\right\} \quad \forall x_2 \in \mathcal{X}_2$$
$$Q_2(x_1) = \min\left\{\hat{x}_2 \in \mathcal{X}_2 : r^1 - \Delta_1 V(\hat{x}_2, x_1) \geq r^2 - \Delta_2 V(\hat{x}_2, x_1)\right\} \quad \forall x_1 \in \mathcal{X}_1.$$

Lemma 14. *Let V be concave and submodular and $x \in \mathbb{Z}_+^2$. Then,*

(i) $x_1 \geq Q_1(x_2)$ *implies* $r^1 - \Delta_1 V(x_1, x_2) < r^2 - \Delta_2 V(x_1, x_2)$,

(ii) $x_2 \geq Q_2(x_1)$ *implies* $r^1 - \Delta_1 V(x_1, x_2) \geq r^2 - \Delta_2 V(x_1, x_2)$,

(iii) $x_1 < Q_1(x_2)$ *if and only if* $x_2 \geq Q_2(x_1)$.

Proof. (i) By definition of Q_1, we have $\Delta_2 V(Q_1(x_2), x_2) - \Delta_1 V(Q_1(x_2), x_2) < r^2 - r^1$. Subconcavity of V implies $\Delta_2 V(x_1, x_2) - \Delta_1 V(x_1, x_2)$ is non-increasing in x_1. Therefore,

$$\Delta_2 V(x_1, x_2) - \Delta_1 V(x_1, x_2) \leq \Delta_2 V(Q_1(x_2), x_2) - \Delta_1 V(Q_1(x_2), x_2) < r^2 - r^1 \quad \forall x_1 \geq Q_1(x_2),$$

which shows (i).
(ii) By definition of Q_2, we have $\Delta_2 V(x_1, Q_2(x_1)) - \Delta_1 V(x_1, Q_2(x_1)) \geq r^2 - r^1$. Subconcavity of V implies $\Delta_2 V(x_1, x_2) - \Delta_1 V(x_1, x_2)$ is non-decreasing in x_2. Therefore,

$$\Delta_2 V(x_1, x_2) - \Delta_1 V(x_1, x_2) \geq \Delta_2 V(x_1, Q_2(x_1)) - \Delta_1 V(x_1, Q_2(x_1)) \geq r^2 - r^1 \quad \forall x_2 \geq Q_2(x_1),$$

which shows (ii).
(iii) The equivalence is a direct consequence of (i) and (ii). □

A direct consequence of Lemma 14 is the following theorem.

Theorem 3. *Let the switching curves $Q_1 = Q_{1jt}$ and $Q_2 = Q_{2jt}$ be defined with $V = V_{t-1}$ and $r = r_j$ for products $j \in \mathcal{J}$ and periods $t \in \mathcal{T}$. Then, the assignment regions in the models (RMFP-s) and (RMFP-d) are*

$$\mathcal{X}_{1jt}^{ass} = \left\{(x_1, x_2)^\top \in \mathcal{X} : x_1 < Q_{1jt}(x_2)\right\} = \left\{(x_1, x_2)^\top \in \mathcal{X} : x_2 \geq Q_{2jt}(x_1)\right\}$$
$$\mathcal{X}_{2jt}^{ass} = \left\{(x_1, x_2)^\top \in \mathcal{X} : x_1 \geq Q_{1jt}(x_2)\right\} = \left\{(x_1, x_2)^\top \in \mathcal{X} : x_2 < Q_{2jt}(x_1)\right\}$$

for all products $j \in \mathcal{J}$ and periods $t \in \mathcal{T}$.

Theorems 2 and 3 establish that both the simple criteria to construct booking paths and switching curves yield the same description of the optimal acceptance and assignment regions.

5.3. Numerical Example

The following numerical example demonstrates the path construction of Algorithm 1 and how optimal booking decisions are derived from a booking path in the static model (RMFP-s). The example assumes some period $t \in \mathcal{T}$ in which we observe demand for product $j = j(t)$ that is flexible having $r_j^1 = 20$ and $r_j^2 = 40$. The total seat capacity of both flights is assumed to be $c_1 = c_2 = 10$ and we set the overbooking penalty $\bar{r} = \infty$.

$V_{t-1}(x)$ \ x_1 / x_2	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
10	190.2	177.0	159.3	139.9	120.0	100.0	80.0	60.0	40.0	20.0	0.0
9	230.2	217.0	199.3	179.9	160.0	140.0	120.0	100.0	80.0	60.0	40.0
8	270.2	256.9	239.3	219.9	200.0	180.0	160.0	140.0	120.0	100.0	80.0
7	310.0	296.8	279.2	259.8	240.0	220.0	200.0	180.0	160.0	140.0	120.0
6	348.5	335.8	318.7	299.7	279.9	260.0	240.0	220.0	200.0	180.0	160.0
5	382.4	372.4	357.2	339.0	319.7	299.9	280.0	260.0	240.0	220.0	200.0
4	407.3	404.2	393.0	377.4	359.0	339.7	319.9	300.0	280.0	260.0	240.0
3	428.2	427.6	424.4	413.0	397.3	378.9	359.6	339.7	319.8	299.8	279.8
2	448.3	448.2	447.6	444.3	432.8	416.8	398.1	378.6	358.6	338.6	318.6
1	468.0	468.0	467.8	467.0	463.1	450.4	433.3	413.9	393.9	373.9	353.9
0	485.3	485.3	485.2	484.7	482.3	475.6	459.7	440.4	420.5	400.5	380.5

Table 3: Revenue-to-go $V_{t-1}(x)$

$r^1(x)$ \ x_1 / x_2	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
10	6.7	2.4	0.6	0.1	0.0	0.0	0.0	0.0	0.0	0.0	$-\infty$
9	6.7	2.4	0.6	0.1	0.0	0.0	0.0	0.0	0.0	0.0	$-\infty$
8	6.7	2.4	0.6	0.1	0.0	0.0	0.0	0.0	0.0	0.0	$-\infty$
7	6.8	2.4	0.6	0.1	0.0	0.0	0.0	0.0	0.0	0.0	$-\infty$
6	7.3	2.9	1.0	0.2	0.1	0.0	0.0	0.0	0.0	0.0	$-\infty$
5	10.0	4.8	1.8	0.8	0.2	0.1	0.0	0.0	0.0	0.0	$-\infty$
4	16.9	8.8	4.4	1.6	0.7	0.2	0.1	0.0	0.0	0.0	$-\infty$
3	19.4	16.8	8.6	4.3	1.6	0.7	0.1	0.0	0.0	0.0	$-\infty$
2	19.9	19.4	16.7	8.4	4.1	1.3	0.4	0.1	0.0	0.0	$-\infty$
1	20.0	19.9	19.1	16.1	7.3	2.9	0.6	0.1	0.0	0.0	$-\infty$
0	20.0	19.9	19.4	17.6	13.3	4.1	0.7	0.1	0.0	0.0	$-\infty$

(a) Allocating flight 1 seats

$r^2(x)$ \ x_1 / x_2	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
10	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.2	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	1.5	1.0	0.5	0.2	0.1	0.0	0.0	0.0	0.0	0.0	0.0
5	6.1	3.4	1.5	0.7	0.2	0.1	0.0	0.0	0.0	0.0	0.0
4	15.1	8.2	4.2	1.6	0.7	0.2	0.1	0.0	0.0	0.0	0.0
3	19.1	16.6	8.6	4.4	1.6	0.8	0.3	0.2	0.2	0.2	0.2
2	19.9	19.4	16.8	8.7	4.6	2.1	1.5	1.2	1.1	1.1	1.1
1	20.3	20.2	19.8	17.4	9.7	6.4	4.9	4.7	4.7	4.7	4.7
0	22.7	22.6	22.6	22.3	20.8	14.8	13.6	13.5	13.5	13.5	13.5

(b) Allocating flight 2 seats

Table 4: Marginal revenue increases $r^1(x)$ and $r^2(x)$ in the example

Using the values $V_{t-1}(x)$ as given in Table 3, we computed the marginal revenue increases of serving one unit of demand $r^1(x)$ and $r^2(x)$, see Table 4. With this information, the indices $i(l)$ for $l = 1$ can be chosen for every state $x \in \mathcal{X}$ as described in Algorithm 1. Table 5 shows in which state we choose to book a single request on flight 1 (1), on flight 2 (2), or to reject it ($-1/-2$).

Figure 1 visualizes the assignment regions $\mathcal{X}_{1jt}^{ass}$ and $\mathcal{X}_{2jt}^{ass}$ and the rejection region $\mathcal{X}_{jt}^{rej} = \mathcal{X} \setminus \mathcal{X}_{jt}^{acc}$. Moreover, it indicates the optimal booking path which results from applying Algorithm 1 in state $x = (0,1)^\top$. The path starts in $x_{(0)} = x$ and ends in $x_{(L)} = x_{(13)} = (8,6)^\top$.

If we further assume that $d = 6$ requests for product j have been observed, it follows from (19) that the optimal booking decision is $u_t^* = (3,3)^\top$ leading to the state $x_{(\ell)} = (3,4)^\top$. Obviously, the marginal values $r^{i(l)}(x_{(l)}) = r_j^{i(l)} - \Delta_{i(l)} V_{t-1}(x_{(l-1)})$ along this path are nonincreasing in l. In general, the result

$$r_j^{i(l)} - \Delta_{i(l)} V_{t-1}(x_{(l-1)}) \geq r_j^{i(l+1)} - \Delta_{i(l+1)} V_{t-1}(x_{(l)}) \quad \forall l \in \{1, 2, \ldots, L-1\} \tag{21}$$

holds for every optimal booking path. It is another consequence of multimodularity and follows by distinguishing two cases:

Case $i(l) = i(l+1)$: It follows $r_j^{i(l)} = r_j^{i(l+1)}$ and with the definition $i = i(l)$ we have $x_{(l)} = x_{(l-1)} + e_i$. Then, (21) is equivalent to $\Delta_i V_{t-1}(x_{(l-1)}) \leq \Delta_i V_{t-1}(x_{(l-1)} + e_i)$, which is concavity of V_{t-1}.

Case $i(l) \neq i(l+1)$: Assume $i(l) = 1$ and $i(l) = 2$. Then,

$$\begin{aligned} & r_j^{i(l)} - \Delta_{i(l)} V_{t-1}(x_{(l-1)}) = r_j^1 - \Delta_1 V_{t-1}(x_{(l-1)}) \\ \geq\ & r_j^2 - \Delta_2 V_{t-1}(x_{(l-1)}) \geq r_j^2 - \Delta_2 V_{t-1}(x_{(l-1)} + e_1) = r_j^{i(l+1)} - \Delta_{i(l+1)} V_{t-1}(x_{(l)}), \end{aligned}$$

where the first inequality results from the definition of $i(l)$ and the second inequality is submodularity of V_{t-1}. The reverse case with $i(l) = 2$ and $i(l+1) = 1$ can be shown in the same manner.

$i(l)$ \ x_1 / x_2	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
10	1	1	1	1	1	1	-1	-1	-1	-1	-2
9	1	1	1	1	1	1	-1	-1	-1	-1	-2
8	1	1	1	1	1	1	-1	-1	-1	-1	-2
7	1	1	1	1	1	1	1	-1	-1	-1	-2
6	1	1	1	1	1	1	1	1	-1	-1	-2
5	1	1	1	1	1	1	2	2	2	2	2
4	1	1	1	1	2	2	2	2	2	2	2
3	1	1	1	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	2	2	2	2	2	2	2	2	2	2	2
0	2	2	2	2	2	2	2	2	2	2	2

Table 5: Choosing $i(l)$ for $l = 1$ in the example

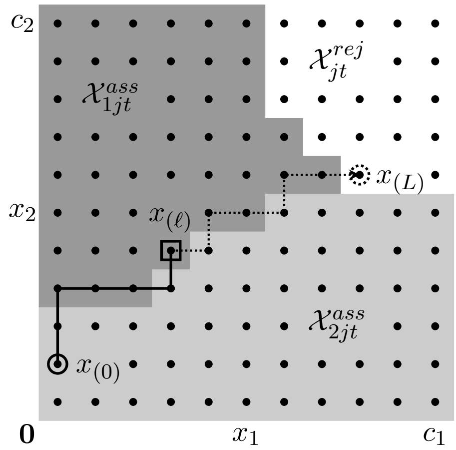

Figure 1: Optimal booking path with $\ell = \min\{d, L\} = \min\{6, 13\}$ in the example

6. Conclusions

In this paper, we extended Lautenbacher and Stidham's (1999) omnibus model to the case with two flights and multiple specific and flexible/opaque products. Our model therefore allows for dynamic as well as static demand distributions. For the latter, the structure of optimal booking control was unknown. Moreover, our model is more general than the flexible-product model by Chen *et al.* (2010) as we can cope with opaque products which generally have lower revenues relative to their constituting specific products. Actually, our analysis becomes simpler compared to the one in Chen *et al.* (2010) and earlier works, since we can uniformly manage specific, flexible, and opaque products by just defining revenues properly. We conducted a comprehensive analysis providing a unifying view on the structure of optimal booking policies under dynamic and static demand. Our paper is the first completely self-contained analysis of important monotonicity properties of the value function (concavity, submodularity, and subconcavity) in both cases. We introduced a new reformulation of the static model which traces the structural analysis back to the dynamic case.

Moreover, we are able to characterize optimal booking policies for the extended omnibus model by means of optimal booking paths exploiting the proven monotonicity properties. We have shown that our path-based policy is an alternative to the switching curve-based policy of Chen *et al.* (2010). Implementing either booking paths or switching curves ends up with an overall computational complexity of $\mathcal{O}\left(n\tau C^2\right)$ in the dynamic and $\mathcal{O}\left(nC^3\right)$ in the static case (C is the largest available seat capacity). Even though there is no computational reason to prefer one optimal policy over the other, we showed that two simple criteria are at the heart of optimal booking control.

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Appendix

A. Submodularity and subconcavity of the single-request operator

In this section, we prove that the single-request operator H^1 defined in (5) preserves multimodularity, as required for the proof of Lemma 7. We follow our discussion of Section 3.3 and examine the double-mode operator H defined in (9) in order to ease the exposition without loss of generality. Our case-by-case analysis is similar to the procedure of Zhuang and Li (2010).

Lemma 15. *If f is multimodular, then the operator $H(f)$ is submodular.*

Proof. By part 3. of Lemma 4, $H(f)$ is submodular if

$$H(f)(x) - H(f)(x+e_2) \leq H(f)(x+e_1) - H(f)(x+e_1+e_2). \tag{22}$$

In what follows, we construct a case for each pair of values for $H(f)(x)$ and $H(f)(x+e_1+e_2)$. Each case is referred to as a pair of the identifiers shown in the first and third column of Table 6a. Note that this procedure requires at most 3^2 cases to check (Zhuang and Li, 2010).

By assumption, f is multimodular and hence implies for all $x \in \mathbb{Z}_+^2$ that

$$f(x) - f(x+e_2) \leq f(x+e_1) - f(x+e_1+e_2) \tag{23}$$
$$f(x) - f(x+e_1) \leq f(x+e_1) - f(x+2e_1) \tag{24}$$
$$f(x) - f(x+e_2) \leq f(x+e_2) - f(x+2e_2), \tag{25}$$

where the first inequality is the submodularity of f (see also part 3. of Lemma 4). Both inequalities (24) and (25) follow from the concavity of f. Starting with case (3.a), note that

$$\begin{aligned} f(x) &- H(f)(x+e_2) \\ &\leq f(x) - f(x+e_2) \leq f(x+e_1) - f(x+e_1+e_2) \\ &\leq r^1 + f(x+2e_1) - f(x+2e_1+e_2) - r^1 \\ &\leq H(f)(x+e_1) - f(x+2e_1+e_2) - r^1, \end{aligned} \tag{26}$$

where the first inequality is true because $H(f)(x+e_2) \geq f(x+e_2)$, the second and third inequalities follow from (23), and the last inequality holds because $H(f)(x+e_1) \geq r^1 + f(x+2e_1)$. Using the inequalities (23) and (25), we get case (3.b):

$$\begin{aligned} f(x) &- H(f)(x+e_2) \\ &\leq f(x) - f(x+e_2) \leq f(x+e_1) - f(x+e_1+e_2) \\ &\leq r^2 + f(x+e_1+e_2) - r^2 - f(x+e_1+2e_2) \\ &\leq H(f)(x+e_1) - r^2 - f(x+e_1+2e_2). \end{aligned} \tag{27}$$

The case (3.c) can be established similarly with inequality (23), i.e., we obtain

$$f(x) - H(f)(x+e_2) \leq H(f)(x+e_1) - f(x+e_1+e_2). \tag{28}$$

Lemma 2 of Zhuang and Li (2010) allows us to combine the results (26), (27), and (28) to

$$f(x) - H(f)(x+e_2) \leq H(f)(x+e_1) - H(f)(x+e_1+e_2). \tag{29}$$

Likewise, the case (1.a) is constructed using (23), while the case (1.b) requires (25). Noting that case (1.c) is trivial because $r^1 + f(x+e_1) - r^1 - f(x+e_1+e_2) = f(x+e_1) - f(x+e_1+e_2)$, we can conclude from the last three cases that

$$r^1 + f(x+e_1) - H(f)(x+e_2) \leq H(f)(x+e_1) - H(f)(x+e_1+e_2). \tag{30}$$

The case (2.c) is trivial because $r^2+f(x+e_2)-f(x+e_2) = r^2+f(x+e_1+e_2)-f(x+e_1+e_2)$. The cases (2.a) and (2.b) are derived using (24) and (23), respectively. Hence,

$$r^2 + f(x+e_2) - H(f)(x+e_2) \leq H(f)(x+e_1) - H(f)(x+e_1+e_2). \tag{31}$$

Finally, combining (29), (30), and (31) yields the result (22) and completes the proof. □

$H(f)(x)$		$H(f)(x+e_1+e_2)$	
ID	Value	ID	Value
1	$r^1 + f(x+e_1)$	a	$r^1 + f(x+2e_1+e_2)$
2	$r^2 + f(x+e_2)$	b	$r^2 + f(x+e_1+2e_2)$
3	$f(x)$	c	$f(x+e_1+e_2)$

(a) Cases of Lemma 15

$H(f)(x+e_1)$		$H(f)(x+2e_2)$	
ID	Value	ID	Value
4	$r^1 + f(x+2e_1)$	d	$r^1 + f(x+e_1+2e_2)$
5	$r^2 + f(x+e_1+e_2)$	e	$r^2 + f(x+3e_2)$
6	$f(x+e_1)$	f	$f(x+2e_2)$

(b) Cases of Lemma 16

Table 6: Definition of identifiers (ID) for the cases of Lemmata 15 and 16

Lemma 16. *If f is multimodular, then the operator $H(f)$ is subconcave.*

Proof. By part 3. of Lemma 5, $H(f)$ is subconcave if

$$H(f)(x+e_1) - H(f)(x+e_2) \leq H(f)(x+e_1+e_2) - H(f)(x+2e_2). \tag{32}$$

We construct the subsequent cases by considering $H(f)(x+e_1)$ and $H(f)(x+2e_2)$, where each case is referred to as a pair of the identifiers depicted in columns one and three of Table 6b. Note that f is multimodular by assumption which implies for all $x \in \mathbb{Z}_+^2$:

$$f(x+e_1) - f(x+e_2) \leq f(x+e_1+e_2) - f(x+2e_2) \tag{33}$$
$$f(x+e_1) - f(x+e_2) \geq f(x+2e_1) - f(x+e_1+e_2) \tag{34}$$
$$f(x) - f(x+e_2) \leq f(x+e_2) - f(x+2e_2), \tag{35}$$

where the first and second inequalities follow from the subconcavity of f in combination with part 3. and 4. of Lemma 5, respectively. The inequality (35) is the definition of concavity in x_2.

Starting with case (4.f), note that

$$\begin{aligned} r^1 + f(x+2e_1) - H(f)(x+e_2) &\leq r^1 + f(x+2e_1) - r^1 - f(x+e_1+e_2) \\ &\leq f(x+e_1+e_2) - f(x+2e_2) \leq H(f)(x+e_1+e_2) - f(x+2e_2), \end{aligned} \tag{36}$$

where the first and last inequalities are true because $H(f)(x+e_2) \geq r^1+f(x+e_1+e_2)$ and $H(f)(x+e_1+e_2) \geq f(x+e_1+e_2)$, respectively, and the second inequality holds by (33) and (34). Likewise, we obtain case (4.d) using (33) and case (4.e) using (33) and (34). Hence, we can conclude that

$$r^1 + f(x+2e_1) - H(f)(x+e_2) \leq H(f)(x+e_1+e_2) - H(f)(x+2e_2). \tag{37}$$

The cases (5.d) and (5.f) are readily available, while case (5.e) is achieved by applying inequality (34). The cases (5.d)-(5.f) lead to

$$r^2 + f(x + e_1 + e_2) - H(f)(x + e_2) \leq H(f)(x + e_1 + e_2) - H(f)(x + 2e_2). \tag{38}$$

The cases (6.e) and (6.f) are obtained with (33) and case (6.d) with (35), and together they imply

$$f(x + e_1) - H(f)(x + e_2) \leq H(f)(x + e_1 + e_2) - H(f)(x + 2e_2). \tag{39}$$

Finally, we achieve the result (32) by combining (37), (38), and (39). This completes the proof. □

Remark 1. Even though a multimodular function is the precondition of both Lemmata 15 and 16, the cases of Lemma 15 are in fact established by using only concavity and submodularity. Similarly, the minimum requirements for the function f of Lemma 16 are in fact concavity and subconcavity.