Online Supplement for "The Shortest-Path Problem with Resource Constraints and k-Cycle Elimination for $k \geq 3$"

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The paper "The Shortest-Path Problem with Resource Constraints and k-Cycle Elimination for $k \geq 3$" mainly focuses on the theoretical aspects of the shortest-path problem with resource constraints (SPPRC) and cycle elimination. The three examples of applying SPPRC-k-cyc to solve VRPTW subproblems were meant to show the relevance of SPPRC-k-cyc and not meant as a detailed computational study on VRPTW.

The online supplement can be vied as a complement to the theoretical work presented in the paper. It gives a detailed analysis of the proposed method for the well-known Solomon (1987) benchmark problems of VRPTW. The online supplement contains a comparison of lower bounds and computation times for 168 different VRPTW instances ranging from 25 to 100 customers.

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1. Extended Computational Results

We start with a description of Solomon's (1987) benchmark problems, give references to the branch-and-price methodology as well as techniques to speed up and improve the solution process. The main part presents the numerical results and discusses the outcome.

1.1. The Solomon Instances

For the computational study, we have used Solomon's (1987) benchmark problems and the same setup as in Kohl et al. (1999), Larsen (1999), and Rich (1999). There are two series of problems, one with approximately 5 to 10 customers per route (type 1) and a second one with long routes with sometimes more than 25 customers in a route (type 2). Within each series, there are three different types of problems, i.e. C-problems, R-problems, and RC-problems where customers are located in clusters (C), randomly (R), and partly in clusters and partly randomly (RC). Hence, there are six groups of problems referred to as (C1, R1, RC1, C2, R2, RC2) with 56 instances. In addition to these original 100 customer problems, instances with 25 and 50 customers are created by considering only the first 25, resp. 50, customers. This leads to a test suite of 168 instances.

As in most papers on exact methods for Solomon's problems the objective is to minimize the total cost, i.e. the travelled distance. Travel times and distances are rounded with a precision of one position after decimal point, see Kohl et al. (1999, page 111). In contrast to this, papers on VRPTW heuristics use unrounded distances and times, and try to minimize the number of routes as the main objective.

It is generally accepted that VRPTW instances with long routes are much harder to solve than the ones with shorter routes. Nevertheless, there are still seven unsolved Solomon benchmark problems with short routes (r104.100, r108.100, r112.100, rc104.100, rc106.100, rc107.100, rc108.100). As reported in Cordeau et al. (2002), there are 35 unsolved problems in the second set, even with 25 and 50 customers and one clustered instance c204.100.

1.2. Branch-and-Price Solution Methodology for the VRPTW

126 of these 168 Solomon problems could previously be solved to optimality, see Cordeau et al. (2002). As far as we know, all of these successfully solved instances can be solved by column generation techniques. From this point of view, column generation and its integration into a branch-and-bound framework (branch-and-price) seems to be the best method at hand. Nevertheless, the idea of k-cycle elimination can analogously be applied to price-directive decomposition approaches based on Lagrangean relaxation, see e.g. Kallehauge et al. (2001).

For the sake of brevity, we do not give an overview of different models and methodologies for the VRPTW here, but refer the reader to the survey paper Cordeau et al. (2002). General references to column generation or branch-and-price are Wolsey (1998) and Barnhart et al. (1998).

Several techniques to improve a standard branch-and-price approach for VRPTW have been published. We use the following ideas:

- Pre-processing, see e.g. Desrochers et al. (1992): Resource window reduction and arc elimination.

- In order to make the costs c_{ij} and the times t_{ij} fulfill the triangle inequality, an offset of 0.1 is added to all cost coefficients c_{ij} except for the start depot $i = s$. The offset does not change the optimal solution and all results can be substituted back by subtracting $0.1 \cdot n$ from the objective, see also Kohl et al. (1999).

- f-path cuts: The basic idea is to integrate cutting plane methods into the column generation technique (sometimes called 'branch-and-price-and-cut'). f-path cuts are valid inequalities for the VRPTW which are added to the RMP when a violated inequality is detected. 2-path cuts have been identified as one key approach to improve the column generation lower bound, see e.g. Kohl (1995), Kohl et al. (1999), and Rich (1999). Their separation subproblem is not trivial but requires the solution of a TSPTW on the corresponding subset of nodes S.

  In our implementation, we use similar techniques as Rich (1999), i.e. Karger's probabilistic algorithm (Karger 1993, Karger and Stein 1993) to identify customer sets S with small flows $x(S)$ and a dynamic programming algorithm for the TSPTW, see e.g. Dumas et al. (1995). We separate 1-path cuts and 2-path cuts only at the root node of the branch-and-bound tree.

- Nearest neighbor networks for *partial pricing*, see Gamache et al. (1999), in Larsen (1999) called *limited subsequence*:

  The pricing problem has to compute at least one new route with negative reduced cost as long as there exists one. By replacing the network G with its smaller ℓ-nearest neighbor network G_ℓ, the pricing problem on G_ℓ is a smaller problem. We work with a hierarchy of three networks G_5, G_{10} and the complete network G. Pricing is firstly done in G_5, if this fails, pricing is done on G_{10} and if this also fails, the complete pricing is done on G. We always add the arcs $(0, i)$ and $(i, n+1)$ which connect the depot with all customers to G_ℓ.

- Branching is first done on the number of vehicles (if the number of routes is fractional) and on arcs secondly. We choose the arc $(i, j) \in A$ with fractional flow x_{ij} which maximizes $c_{ij} \cdot \min(x_{ij}, 1 - x_{ij})$. But there is one exception from this branching rule.

Within the branch-and-bound tree, solving instances with the additional constraint that the number of vehicles (after branching) has to be equal to one is sometimes very hard. We observed that the corresponding nodes of the branch-and-bound tree had huge computation times and sometimes we were not able to solve them (within the given time limit).

In order to overcome this problem we used the following strategy: Whenever a solution of the RMP has a fractional number of vehicles $\#veh$ with $1 < \#veh < 2$ we do not branch on the number of vehicles but on arcs.

In order to keep the number of branch-and-bound nodes to explore as small as possible, we implemented the following rule. Within branch-and-bound a *best-first* node selection strategy is used. It means that among all unsolved nodes we choose one with currently minimum lower bound (note that each son node gets an initial lower bound from its father before that node is solved by column generation).

1.3. Comparison of 2-, 3-, and 4-Cycle Elimination

All computations were performed on a standard PC with Intel Pentium III, 600MHz with 512MB main memory. The algorithm is coded in C++ and the callable library of CPLEX 7.0 CPLEX (1997) is used to solve the restricted master problem (RMP).

1.3.1. Comparison with a Restricted Computing Time of 1 Hour

We start with a detailed analysis of the lower bounds for k-cycle elimination with $k \in \{2, 3, 4\}$. The computation time for each instance is restricted to one hour (=3600 seconds). The Tables 1–3 contain the following information:

- The name of the *instance* is given in the first column.
- The *integrality gap* of the instance is the interval $[lb_1(1) : opt]$ given by the lower bound $lb_1(1)$ computed *without* cycle-elimination (a plain SPPRC subproblem solver) and the objective *opt* of an optimal solution. In case the optimum *opt* is not known, we give a valid upper bound *ub* computed by a heuristic algorithm[1] and mark that entry with ub^*.

[1] Special thanks to Birger Funke who computed the upper bounds for some of the hard Solomon benchmark instances with methods described in Funke (2003).

- Three compound sections are given for $k = 2$, $k = 3$, and $k = 4$, and describe the outcome of the branch-and-price procedure with the SPPRC-k-cyc subproblem solver.

- $lb_1(k)$ is the lower bound implied by the LP-relaxation of the master program before cutting planes are added. For some instances and different values of k we were not able to solve the LP-relaxation of the master program to optimality. This fact is indicated by an entry '–'.

- $lb_2(k)$ is the lower bound after adding 1-path cuts and 2-path cuts. If $lb_1(k)$ and $lb_2(k)$ are identical to $lb_1(k-1)$ and $lb_2(k-1)$ for $k \geq 3$ we do not print the same information again.

- Whenever we are not able to solve the instance to optimality the entry $lb(k)$ gives the computed lower bound at the moment when the computation was stopped (3600s).

- The size of the branch-and-price search is given by the number of *tree* nodes. For instances not solved to optimality, this column indicates the number of the tree nodes evaluated within one hour.

- $T(k)$ gives the time for completing the computation or *TL* (=time limit) when the computation was stopped after 3600s.

A comparison of the detailed results for k-cycle elimination given in the Tables 1–3 can be aggregated to the following characteristic numbers:

How many instances are solved to optimality with $k = 2$?	112
How many instances are solved to optimality with $k = 3$?	117
How many instances are solved to optimality with $k = 4$?	117
How many instances are solved to optimality with $k \in \{2, 3, 4\}$?	124
How often is $k = 3$ optimal but $k = 2$ not optimal?	7
How often is $k = 4$ optimal but $k = 3$ not optimal?	5
How often is $k = 4$ optimal but $k = 2$ not optimal?	11
How often is $k = 2$ optimal but neither $k = 3$ nor $k = 4$ optimal?	2
How often is time $T(3)$ smaller than $T(2)$?	42
How often is time $T(4)$ smaller than $T(3)$?	17
How often is time $T(4)$ smaller than $T(2)$?	41
How often is $T(3)$ or $T(4)$ smaller than $T(2)$?	47

Comparing the results for $k = 3$ and $k = 4$ against $k = 2$ shows that 3-cycle and 4-cycle elimination are successful only for some instances. "Successful" means, that improved

Table 1: Results for Solomon Benchmark Problems with 25 Customers

Instance	Integrality Gap	$lb_1(2)$	$lb_2(2)$	$lb(2)$	Tree	Time	$lb_1(3)$	$lb_2(3)$	$lb(3)$	Tree	Time	$lb_1(4)$	$lb_2(4)$	$lb(4)$	Tree	Time
				k=2					k=3					k=4		
C101.25	191,3	191,3	191,3	opt	1	0,5			opt	1	0,5			opt	1	0,5
C102.25	[189,15 : 190,3]	190,3	190,3	opt	1	0,8			opt	1	0,9			opt	1	1,5
C103.25	[189,15 : 190,3]	190,3	190,3	opt	1	0,9			opt	1	1,8			opt	1	2,2
C104.25	[185,75 : 186,9]	186,9	186,9	opt	1	1,2			opt	1	2,8			opt	1	27,9
C105.25	191,3	191,3	191,3	opt	1	0,4			opt	1	0,4			opt	1	0,4
C106.25	191,3	191,3	191,3	opt	1	0,4			opt	1	0,4			opt	1	0,4
C107.25	191,3	191,3	191,3	opt	1	0,6			opt	1	0,6			opt	1	0,6
C108.25	[190,5 : 191,3]	191,3	191,3	opt	1	0,5			opt	1	0,4			opt	1	0,4
C109.25	[188,46 : 191,3]	191,3	191,3	opt	1	0,6			opt	1	1,4			opt	1	1,8
R101.25	617,1	617,1	617,1	opt	1	0,1			opt	1	0,1			opt	1	0,1
R102.25	[546,333 : 547,1]	546,333	547,1	opt	2	0,4			opt	2	0,4			opt	2	0,5
R103.25	[454,067 : 454,6]	454,6	454,6	opt	1	0,2			opt	1	0,3			opt	1	0,3
R104.25	[414,85 : 416,9]	416,9	416,9	opt	1	0,3			opt	1	0,4			opt	1	0,6
R105.25	530,5	530,5	530,5	opt	1	0,1			opt	1	0,1			opt	1	0,1
R106.25	[457,3 : 465,4]	457,3	465,4	opt	2	1,2			opt	2	1,2			opt	2	1,3
R107.25	[415,125 : 424,3]	422,925	424,3	opt	2	1,5	424,3		opt	1	0,3			opt	1	0,4
R108.25	[389,424 : 397,3]	396,139	396,139	opt	3	5,0	396,72	396,72	opt	3	6,2		397,3	opt	2	2,9
R109.25	[439,425 : 441,3]	441,3	441,3	opt	1	0,2			opt	1	0,2			opt	1	0,2
R110.25	[419,072 : 444,1]	437,3	437,363	opt	27	7,0	438,35	438,8	opt	9	5,0			opt	9	5,1
R111.25	[412,815 : 428,8]	423,788	428,05	opt	6	5,5	427,283	428,8	opt	4	3,1			opt	4	3,3
R112.25	[365,03 : 393]	384,2	385,391	opt	11	13,0	386,94	387,015	opt	9	18,0	387,05	387,486	opt	10	32,2
RC101.25	[390,15 : 461,1]	406,625	461,1	opt	2	1,1			opt	2	1,1			opt	2	1,2
RC102.25	[347,067 : 351,8]	351,8	351,8	opt	1	0,4			opt	1	0,4			opt	1	0,4
RC103.25	[317,538 : 332,8]	332,05	332,05	opt	3	1,6			opt	3	1,8			opt	3	2,0
RC104.25	[299,066 : 306,6]	305,825	305,825	opt	5	2,4			opt	5	3,6			opt	5	13,8
RC105.25	[408,525 : 411,3]	410,95	410,95	opt	3	1,1			opt	3	1,1			opt	3	1,3
RC106.25	[319,661 : 345,5]	342,829	343,2	opt	10	2,3	345,5	345,5	opt	1	0,4			opt	1	0,5
RC107.25	[292,947 : 298,3]	298,3	298,3	opt	1	0,6			opt	1	0,4			opt	1	3,2
RC108.25	[286,565 : 294,5]	293,791	294,5	opt	2	0,7	294,5		opt	1	0,9			opt	1	6,9
C201.25	214,7	214,7	214,7	opt	1	1,8			opt	1	1,8			opt	1	1,8
C202.25	214,7	214,7	214,7	opt	1	1,6			opt	1	2,3			opt	1	7,0
C203.25	[213,775 : 214,7]	214,7	214,7	opt	1	3,7			opt	1	7,9			opt	1	51,7
C204.25	[207,156 : 213,1]	210,992	211,046	opt	11	279,8	211,042		opt	10	392,3	211,9		opt	7	1515,9
C205.25	[196,525 : 214,7]	212,05	214,7	opt	3	2,7	214,7		opt	1	1,4			opt	1	1,4
C206.25	[194,015 : 214,7]	197,7	214,7	opt	4	3,5	213,95		opt	3	2,4	214,7		opt	1	1,7
C207.25	[200,442 : 214,5]	207,981	214,4	opt	14	51,4	211,49		opt	12	36,9	212,65		opt	9	46,5
C208.25	[183,845 : 214,5]	193,28	214,5	opt	6	5,9	213,433		opt	5	4,3	213,55		opt	2	3,0
R201.25	[448,5 : 463,3]	460,1	460,1	opt	3	2,3			opt	3	2,0			opt	3	1,9
R202.25	[374,117 : 410,5]	406,35	408,35	opt	6	4,0	410,5	410,5	opt	1	0,8			opt	1	1,2
R203.25	[337,51 : 391,4]	379,882	381,625	opt	46	55,2	387,15	387,15	opt	11	12,4	391,4	391,4	opt	1	5,8
R204.25	[304,246 : 355]	333,075	335,35	opt	622	3523,7	341,733	342,58	opt	104	385,2	349,1	349,1	opt	35	231,7
R205.25	[365,475 : 393]	381,283	388,45	opt	16	9,7	390,6	390,6	opt	7	7,0			opt	5	6,3
R206.25	[317,98 : 374,4]	363,132	365,908	opt	75	268,4	373,6	373,6	opt	7	5,1			opt	7	11,5
R207.25	[309,609 : 361,6]	347,592	349,741	opt	179	986,1	357,179	358,571	opt	25	52,6	360,05	360,05	opt	11	56,4
R208.25	[295,923 : 328,2]	318,105	318,922	opt	75	1861,9	323,331	323,361	opt	16	363,5	327,06	327,06	opt	9	388,0
R209.25	[327,253 : 370,7]	353,875	358,321	opt	46	72,2	364,05	364,05	opt	9	8,4			opt	9	15,1
R210.25	[340,505 : 404,6]	395,844	397,906	opt	56	109,2	403,6	403,6	opt	3	8,0	404,175	404,175	opt	3	17,6
R211.25	[299,998 : 350,9]	330,14	330,466	opt	695	TL	336,244	336,483	opt	456	1831,4	339,981	339,981	opt	139	876,3
RC201.25	[315,853 : 360,2]	356,65	360,2	opt	2	0,9	360,2		opt	1	0,7			opt	1	0,8
RC202.25	[234,629 : 338]	290,408	312,992	opt	130	950,4	308,033	321,425	opt	7	12,6	314,9	324,15	opt	6	24,4
RC203.25	[185,477 : 326,9]	213,508	260,385	287,54	218	TL	254,55	278,592	311,03	458	TL	282,95	297,675	opt	297	3455,3
RC204.25	[169,544 : 299,7*]	188,213	244,71	257,15	131	TL	215,717	252,467	270,27	139	TL	235,925	266,25	278,29	83	TL
RC205.25	[258,961 : 338]	307,6	320,787	opt	42	57,9	338	338	opt	1	1,5			opt	1	3,1
RC206.25	[198,363 : 324]	249,2	288,978	319,36	300	TL	310	311,15	opt	41	48,0	318,95	318,95	opt	9	12,3
RC207.25	[175,003 : 298,3]	216,67	263,894	284,21	346	TL	257,011	272,65	opt	692	3588,2	264,6	280,4	opt	134	1393,7
RC208.25	[155,514 : 269,1*]	169,671	233,078	243,49	89	TL	179,664	235,401	247,87	145	TL	186,77	238,26	242,33	25	TL

Table 2: Results for Solomon Benchmark Problems with 50 Customers

Instance	Integrality Gap	$lb_1(2)$	$lb_2(2)$	$lb(2)$	Tree	Time	$lb_1(3)$	$lb_2(3)$	$lb(3)$	Tree	Time	$lb_1(4)$	$lb_2(4)$	$lb(4)$	Tree	Time
				k=2					k=3					k=4		
C101.50	362,4	362,4	362,4	opt	1	3,2			opt	1	3,2			opt	1	3,2
C102.50	[360,25 : 361,4]	361,4	361,4	opt	1	7,0			opt	1	7,2			opt	1	15,7
C103.50	[360,25 : 361,4]	361,4	361,4	opt	1	13,4			opt	1	31,5			opt	1	139,9
C104.50	[354,021 : 358]	357,25	358	opt	3	67,2	358		opt	1	161,0			opt	1	2151,6
C105.50	362,4	362,4	362,4	opt	1	4,1			opt	1	4,2			opt	1	4,2
C106.50	362,4	362,4	362,4	opt	1	3,2			opt	1	3,3			opt	1	3,3
C107.50	362,4	362,4	362,4	opt	1	3,9			opt	1	3,9			opt	1	4,0
C108.50	[361,81 : 362,4]	362,4	362,4	opt	1	5,7			opt	1	5,5			opt	1	5,8
C109.50	[359,365 : 362,4]	362,4	362,4	opt	1	5,6			opt	1	12,6			opt	1	19,7
R101.50	[1043,37 : 1044]	1043,37	1044	opt	2	3,1			opt	2	3,2			opt	2	3,1
R102.50	909	909	909	opt	1	1,7			opt	1	1,7			opt	1	1,5
R103.50	[756,117 : 772,9]	765,95	767,3	opt	32	35,3	768,1	769,139	opt	17	33,4	769,056	769,44	opt	16	38,8
R104.50	[608,521 : 625,4]	616,5	620,758	opt	34	212,6	618,373	621,864	opt	24	353,9	619,001	622,496	opt	16	1438,0
R105.50	[890,187 : 899,3]	892,12	893,65	opt	8	11,9			opt	8	11,9			opt	8	11,9
R106.50	[789,433 : 793]	791,367	793	opt	2	8,6			opt	2	8,9			opt	2	9,1
R107.50	[697,767 : 711,1]	704,438	705,88	opt	31	61,5	705,006	706,383	opt	22	55,8	705,321	706,621	opt	22	88,9
R108.50	[578,482 : 617,7]	588,926	595,177	614,7	290	TL	593,852	597,623	615,37	207	TL	594,213	598,152	610,83	26	TL
R109.50	[727,515 : 786,8]	775,096	776,231	opt	197	148,4	775,342	776,62	opt	141	100,9			opt	141	102,2
R110.50	[675,457 : 697]	692,577	694,15	opt	16	26,4	694,014	696,219	opt	17	54,1	694,063	696,444	opt	11	40,1
R111.50	[658,752 : 707,2]	691,812	692,642	opt	210	300,6	693,01	695,312	opt	109	227,3	693,234	695,668	opt	117	334,3
R112.50	[582,715 : 630,2]	607,219	612,36	627,03	693	TL	612,923	615,069	opt	659	3522,8	614,503	616,423	627,51	233	TL
RC101.50	[826,613 : 944]	850,021	944	opt	4	26,4			opt	4	26,0			opt	4	26,1
RC102.50	[706,606 : 822,5]	719,902	813,037	opt	143	303,8	720,419	813,767	opt	155	317,1			opt	160	359,5
RC103.50	[612,24 : 710,9]	643,133	710,667	opt	12	90,0	643,158		opt	11	90,0	643,225		opt	12	150,9
RC104.50	[527,171 : 545,8]	543,75	543,75	opt	19	68,5	545,5	545,5	opt	3	77,8	545,8	545,8	opt	1	465,2
RC105.50	[746,314 : 855,3]	754,443	853,675	opt	19	64,9	756,817	854,86	opt	12	54,8	759,314	855,075	opt	12	58,7
RC106.50	[633,228 : 723,2]	664,433	717,155	opt	50	93,7		718,966	opt	22	50,2		719,996	opt	16	42,7
RC107.50	[570,665 : 642,7]	591,476	632,336	opt	333	877,1	601,227	638,743	opt	23	167,5	602,014	639,53	opt	19	609,1
RC108.50	[525,12 : 598,1]	538,957	590,469	opt	15	527,8	540,48	593,816	opt	9	666,6		594,517	594,52	6	TL
C201.50	360,2	360,2	360,2	opt	1	16,8			opt	1	16,9			opt	1	17,1
C202.50	360,2	360,2	360,2	opt	1	196,8			opt	1	233,6			opt	1	771,8
C203.50	359,8	359,8	359,8	opt	1	228,4			opt	1	859,4			opt	1	2655,4
C204.50	[347,392 : 350,1]	350,1	350,1	opt	1	1159,4	–	–	–		TL	–	–	–		TL
C205.50	[341,762 : 359,8]	357,35	359	opt	4	222,4	359,8	359,8	opt	1	36,5			opt	1	38,7
C206.50	[338,518 : 359,8]	344,2	359	opt	4	567,4	359,8	359,8	opt	1	86,5			opt	1	108,9
C207.50	[348,625 : 359,6]	356,269	359,6	opt	6	274,0	358,931		opt	4	386,8	359,6		opt	1	1512,5
C208.50	[331,27 : 350,5]	340,425	350,5	opt	2	138,5	350,5		opt	1	307,6			opt	1	662,5
R201.50	[754,098 : 791,9]	788,425	791,9	opt	2	13,3	791,9		opt	1	6,4			opt	1	7,7
R202.50	[637,594 : 698,5]	692,737	696,525	opt	8	66,1	698,5	698,5	opt	1	17,3			opt	1	47,8
R203.50	[538,526 : 605,3]	590,93	593,43	603,98	134	TL	598,35	598,35	opt	23	470,4	598,583	598,583	opt	5	665,9
R204.50	[440,989 : 506,4]	474,562	482,324	487,05	35	TL	485,031	486,822	493,33	37	TL	500,072	500,075	501,8	13	TL
R205.50	[595,553 : 690,1]	666,604	672,351	682,1	241	TL	678,329	678,329	opt	295	3498,4	682,786	682,786	opt	137	1091,4
R206.50	[530,574 : 632,4]	609,59	611,363	620,54	109	TL	618,777	618,777	627,34	142	TL	621,277	621,277	628,31	77	TL
R207.50	[467,314 : 575,5*]	538,762	544,324	551,33	50	TL	554,136	554,136	560,27	66	TL	556,809	557,396	560,47	24	TL
R208.50	[??? : 487,7*]	462,406	471,563	472,65	15	TL	472,224	473,23	475,05	15	TL	474,706	474,752	475,32	4	TL
R209.50	[535,279 : 600,6]	582,877	588,413	597,81	170	TL	591,844	594,097	opt	59	705,3	599,813	599,813	opt	7	255,4
R210.50	[532,119 : 645,6]	624,155	624,162	633,54	160	TL	630,673	630,675	641,03	165	TL	633,03	633,03	641,79	78	TL
R211.50	[457,426 : 535,5]	507,95	512,558	522	146	TL	519,989	520,377	530,59	179	TL	525,98	525,98	531,69	62	TL
RC201.50	[530,547 : 684,8]	670,15	681,983	opt	15	51,5	680,44	682,867	opt	8	30,6	683,133	683,133	opt	5	25,7
RC202.50	[416,67 : 613,6]	494,608	546,359	557,44	88	TL	556,3	585,2	599,28	216	TL	589,55	604,417	opt	28	500,3
RC203.50	[326,429 : 555,3]	408,496	490,122	490,79	22	TL	448,117	509,463	516,14	38	TL	489,55	529,95	530,2	12	TL
RC204.50	[??? : 444,2*]	314,447	406,094	406,09	10	TL	332,616	411,691	411,69	6	TL	–	–	–		TL
RC205.50	[481,604 : 630,2]	541,592	581,528	592,13	65	TL	589,313	612,683	opt	62	729,9	621,6	630,2	opt	5	82,4
RC206.50	[363,874 : 610]	441,336	532,959	543,99	46	TL	511,458	557,814	578,64	141	TL	594,483	597,05	opt	62	934,9
RC207.50	[333,105 : 558,6*]	390,822	469,64	480,35	36	TL	415,942	479,612	492,94	38	TL	456,133	504,835	517,1	28	TL
RC208.50	[264,761 : 479,2*]	316,325	416,051	418,52	19	TL	331,562	422,248	424,86	15	TL	–	–	–		TL

Table 3: Results for Solomon Benchmark Problems with 100 Customers

Instance	Integrality Gap	$lb_1(2)$	$lb_2(2)$	$lb(2)$ k=2	Tree	Time	$lb_1(3)$	$lb_2(3)$	$lb(3)$ k=3	Tree	Time	$lb_1(4)$	$lb_2(4)$	$lb(4)$ k=4	Tree	Time
C101	827,3	827,3	827,3	opt	1	30,4			opt	1	30,0			opt	1	30,4
C102	827,3	827,3	827,3	opt	1	56,9			opt	1	77,0			opt	1	216,0
C103	826,3	826,3	826,3	opt	1	130,4			opt	1	339,5			opt	1	2596,1
C104	[822,389 : 822,9]	822,9	822,9	opt	1	358,5			opt	1	1533,8	–	–	–		TL
C105	827,3	827,3	827,3	opt	1	35,6			opt	1	36,1			opt	1	36,7
C106	827,3	827,3	827,3	opt	1	35,0			opt	1	39,7			opt	1	40,8
C107	827,3	827,3	827,3	opt	1	34,5			opt	1	35,1			opt	1	35,7
C108	[820,988 : 827,3]	827,3	827,3	opt	1	49,0			opt	1	60,3			opt	1	64,4
C109	[817,485 : 827,3]	825,64	827,3	opt	2	190,2	827,3		opt	1	163,0			opt	1	254,5
R101	[1631,15 : 1637,7]	1631,15	1634	opt	12	115,8			opt	12	117,6			opt	12	118,2
R102	1466,6	1466,6	1466,6	opt	1	21,8			opt	1	24,2			opt	1	28,7
R103	[1203,24 : 1208,7]	1206,31	1206,42	opt	26	446,1	1206,54	1206,57	opt	20	499,8			opt	20	830,4
R104	[937,062 : 971,5]	949,103	951,101	959,08	77	TL	955,29	955,705	961,18	44	TL	–	–	–		TL
R105	[1341,19 : 1355,3]	1346,14	1349,32	opt	47	358,3			opt	47	361,7			opt	47	365,5
R106	[1212,27 : 1234,6]	1226,44	1227,4	opt	204	2132,8			opt	206	2182,0			opt	206	2373,0
R107	[1036,96 : 1064,6]	1051,84	1052,94	1063,4	182	TL	1052,23	1053,4	1063,2	140	TL	1052,88	1053,91	1061,6	58	TL
R108	[891,646 : 939*]	907,162	910,617	918,97	94	TL	911,493	913,449	919,89	36	TL	912,338	913,845	913,85	3	TL
R109	[1097,46 : 1146,9]	1130,59	1133,16	1139,8	307	TL	1134,26	1134,93	1143,1	367	TL	1134,28	1135,06	1143,3	349	TL
R110	[1021,33 : 1068]	1048,48	1049,94	1058,8	203	TL	1052,52	1052,81	1061	157	TL	1055,33	1055,75	1061,5	101	TL
R111	[1005,88 : 1048,7]	1032,03	1032,07	1044	241	TL	1034,17	1034,23	1044,4	173	TL	1034,25	1034,35	1042,7	82	TL
R112	[892,545 : 960,5*]	919,192	922,412	928,55	91	TL	923,575	925,811	928,37	11	TL	–	–	–		TL
RC101	[1567,45 : 1619,8]	1584,09	1617,4	opt	14	462,9			opt	14	464,7			opt	14	469,1
RC102	[1380,21 : 1457,4]	1403,65	1439,55	1449	161	TL	1404,94	1440,66	1450,5	164	TL	1405,05		1450,5	154	TL
RC103	[1170,32 : 1258]	1218,5	1241,72	1247,8	75	TL	1221,73	1242,33	1248,2	70	TL	1223,53	1242,93	1247,5	33	TL
RC104	[1052,55 : 1132,3]	1094,33	1112,35	1113	9	TL	1097,17	1114,33	1114,9	6	TL	–	–	–		TL
RC105	[1453,89 : 1513,7]	1471,16	1509,8	opt	25	899,0	1471,83		opt	17	921,9	1471,93		opt	17	921,7
RC106	[1248,96 : 1373,9*]	1308,78	1333,38	1346,6	237	TL	1317,17	1341,86	1353,2	197	TL	1318,8	1343,08	1354,8	228	TL
RC107	[1117,32 : 1207,8]	1170,69	1186,43	1195,3	151	TL	1180,96	1195	1202,5	158	TL	1182,5	1195,38	1201,8	78	TL
RC108	[1035,93 : 1114,2]	1063,01	1097,26	1101,5	66	TL	1066,77	1100,5	1103,9	26	TL	–	–	–		TL
C201	589,1	589,1	589,1	opt	1	99,8			opt	1	100,9			opt	1	101,1
C202	589,1	589,1	589,1	opt	1	585,6			opt	1	753,4	–	–	–		TL
C203	[585,767 : 588,7]	588,7	588,7	opt	1	1706,3	–	–	–		TL	–	–	–		TL
C204	[582,226 : 588,1]	–	–	–		TL	–	–	–		TL	–	–	–		TL
C205	[582,362 : 586,4]	586,4	586,4	opt	1	328,3			opt	1	221,9			opt	1	230,6
C206	[576,032 : 586]	585,4	586	opt	2	1263,1	586		opt	1	814,4			opt	1	1322,2
C207	[571,069 : 585,8]	581,969	585,8	opt	5	2068,8	585,131		opt	4	2205,2	–	–	–		TL
C208	[570,461 : 585,8]	581,767	585,8	opt	2	2183,3	585,8		opt	1	2348,8			opt	1	3530,3
R201	[1080,74 : 1143,2]	1136,22	1138,65	1141,4	34	TL	1140	1140	1142,8	31	TL	1140,3	1140,3	opt	35	3537,2
R202	[933,494 : 1038,4*]	–	–	–		TL	–	–	–		TL	–	–	–		TL
R203	[??? : 876,5*]	846,489	847,1	847,1	4	TL	–	–	–		TL	–	–	–		TL
R204	[??? : 738,4*]	–	–	–		TL	–	–	–		TL	–	–	–		TL
R205	[838,435 : 954,2*]	916,572	922,704	926,26	15	TL	930,63	931,168	932,6	8	TL	–	–	–		TL
R206	[??? : 887,5*]	834,57	840,671	840,67	4	TL	–	–	–		TL	–	–	–		TL
R207	[??? : 804,3*]	–	–	–		TL	–	–	–		TL	–	–	–		TL
R208	[??? : 703,5*]	–	–	–		TL	–	–	–		TL	–	–	–		TL
R209	[750,431 : 855,7*]	819,847	823,733	826	13	TL	833,926	834,37	835,87	6	TL	–	–	–		TL
R210	[753,975 : 909,9*]	849,334	855,654	855,65	4	TL	–	–	–		TL	–	–	–		TL
R211	[650,812 : 756,2*]	705,806	710,744	710,74	7	TL	–	–	–		TL	–	–	–		TL
RC201	[1107,04 : 1261,8]	1240,4	1253,49	1256,5	36	TL	1255,35	1255,99	1260,7	44	TL	1255,77	1256,26	opt	39	3620,4
RC202	[880,357 : 1092,3*]	1004,05	1013,69	1018	14	TL	1050,9	1054,99	1059,6	11	TL	–	–	–		TL
RC203	[693,611 : 934,7*]	–	–	–		TL	–	–	–		TL	–	–	–		TL
RC204	[??? : 788,4*]	–	–	–		TL	–	–	–		TL	–	–	–		TL
RC205	[967,076 : 1154]	1055,52	1082,05	1088,7	19	TL	1117,93	1130,5	1134,2	17	TL	–	–	–		TL
RC206	[852,168 : 1051,1*]	952,182	982,059	984,97	12	TL	1005,85	1016,99	1018	9	TL	–	–	–		TL
RC207	[768,03 : 965,9*]	866,258	877,009	877,61	9	TL	886,915	891,895	891,9	5	TL	–	–	–		TL
RC208	[??? : 776,5*]	–	–	–		TL	–	–	–		TL	–	–	–		TL

lower bounds imply significantly smaller branch-and-bound trees and that smaller trees overcompensate the higher effort to solve SPPRC with k-cycle elimination (instead of solving SPPRC without or only with 2-cycle elimination). The success for only some of the instances had to be expected because not all fractional solutions of the RMPs contain routes with cycles.

A comparison of the computing times shows that 47 of the 124 solved instances are solved faster by 3- or 4-cycle elimination than with 2-cycle elimination. Ten instances (rc203.25, rc206.25, rc207.25, r112.50, r203.50, r205.50, r209.50, rc202.50, rc205.50, rc206.50) could only be solved by the new algorithms ($k = 3, 4$) within one hour computing time. Five of them (r203.50, r209.50, rc202.50, rc205.50, rc206.50) were previously unsolved. The previously unsolved instance r204.25 is now solved to optimality by all of the three algorithms ($k = 2, 3, 4$). This is due to the modified branching rule.

There are two instances (r204.50, c203.100) for which the new algorithms ($k = 3, 4$) failed to compute the optimal solution, which are solved by using 2-cycle elimination. It seems that k-cycle elimination with $k \geq 3$ is more successful for instances with long routes than for instances with short routes and more useful when customers are not clustered.

For some instances, especially those with long routes, the difference in the lower bounds $lb_1(k)$ and $lb_2(k)$ and computation times are impressive. We point out some examples.

RC205.50 For the instance rc205.50 the lower bounds rise from $lb_1(1) = 481.604$ to $lb_1(2) = 541,592$, $lb_1(3) = 589,313$, $lb_1(4) = 621.6$ and with cutting planes from $lb_2(2) = 581.528$ to $lb_2(3) = 612.683$, $lb_2(4) = 630.2$. At the same time the computing times fall from $T(2) > 3600s$ (unsolved) to $T(3) = 729.9s$ and $T(4) = 82.4s$. The optimal solution with $opt = 631.0$ is very close to the solution corresponding to 4-cycle elimination and cutting planes.

RC202.25 Another interesting example is the instance rc202.25. The size of the branch-and-bound tree falls from 130 node for $k = 2$ to 7 resp. 6 nodes for $k = 3$ or $k = 4$. From this, a speedup of factor $T(3)/T(2) \approx 75$ for 3-cycle elimination or of factor $T(4)/T(2) \approx 39$ for 4-cycle elimination results.

Visualization In order to illustrate the difference in the computed lower bounds $lb_1(k)$ and $lb_2(k)$ with respect to the values of $k \in \{2, 3, 4\}$, it makes sense to visualize the portion

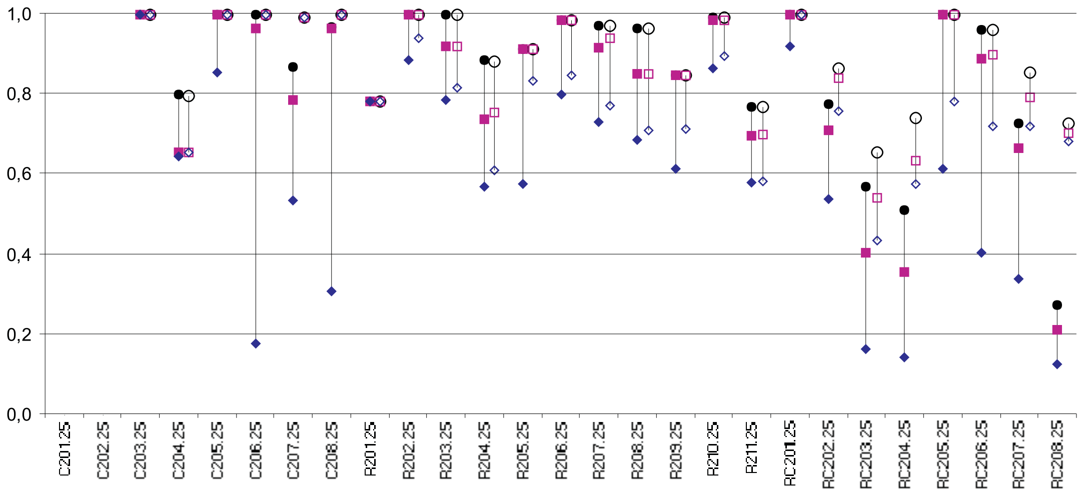

Figure 1: Results for the Solomon Problems with 25 Customers and *Long* Routes

of the integrality gap that has been closed by applying k-cycle elimination (the same measure has been used in Kohl et al. (1999) to show the effectiveness of the f-path cuts). For different values of $k > 0$ the numbers

$$\frac{lb_1(k) - lb_1(1)}{opt - lb_1(1)} \quad \text{and} \quad \frac{lb_2(k) - lb_1(1)}{opt - lb_1(1)}$$

describe the relative portion of the integrality gap that has been closed by k-cycle elimination with/without using additional f-path cuts. Obviously, these numbers only exist when the denominator is positive, i.e. the corresponding instance has an integrality gap $opt - lb_1(1) > 0$. If the optimal solution is not known we use a best known upper bound ub^* instead of opt. The Figures 1–4 depict the closed portion of the integrality gap for the problems with 25 customers and long routes, all instances with 50 customers, and the instances with 100 customers and short routes. Each figure shows the portion of the integrality gap $[lb_1(1), opt]$ resp. $[lb_1(1), ub^*]$ which has been closed by applying k-cycle elimination. For instances with a proper positive integrality gap (up to) six values $lb_1(k)$, $lb_2(k)$ for $k \in \{2, 3, 4\}$ are given. The three lower bounds $lb_1(2), lb_1(3), lb_1(4)$ obtained without cutting planes as well as the three values $lb_1(2), lb_1(3), lb_1(4)$ valid after adding f-path cuts are displayed in a line. Values for $k = 2$ are marked with $\blacklozenge, \lozenge$, for $k = 3$ with $\blacksquare, \square$, and for $k = 4$ with $\bullet$, $\circ$. The remaining groups of instances are not presented because there is either almost always no integrality gap (series C1, R1, and RC1 with 25 customers) or we were not able to compute many of the bounds within one hour (series C2, R2, and RC2 with 100 customers).

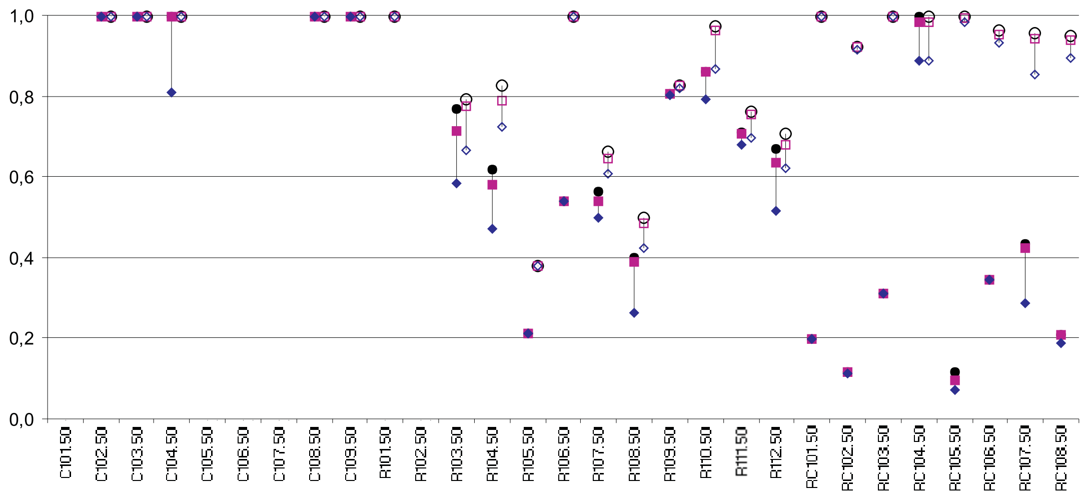

Figure 2: Results for the Solomon Problems with 50 Customers and *Short* Routes

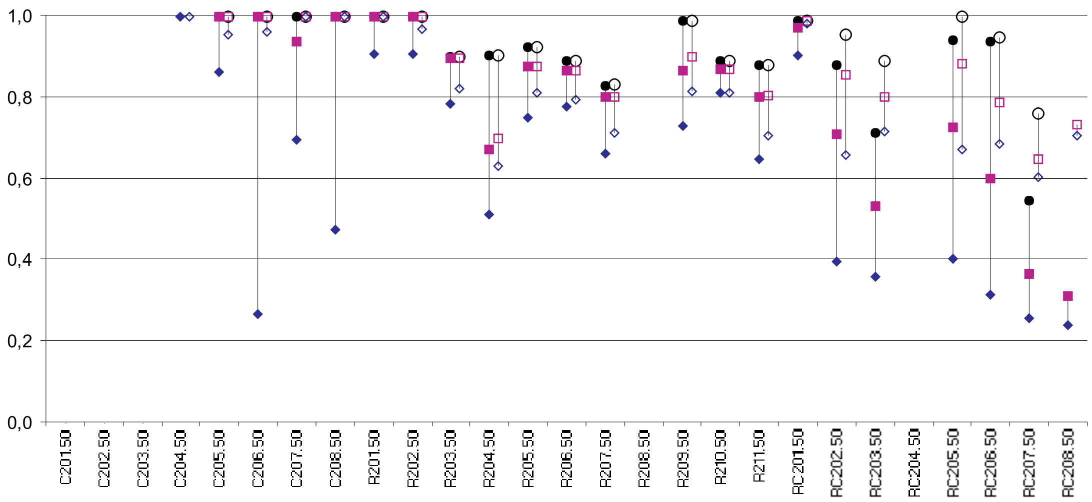

Figure 3: Results for the Solomon Problems with 50 Customers and *Long* Routes

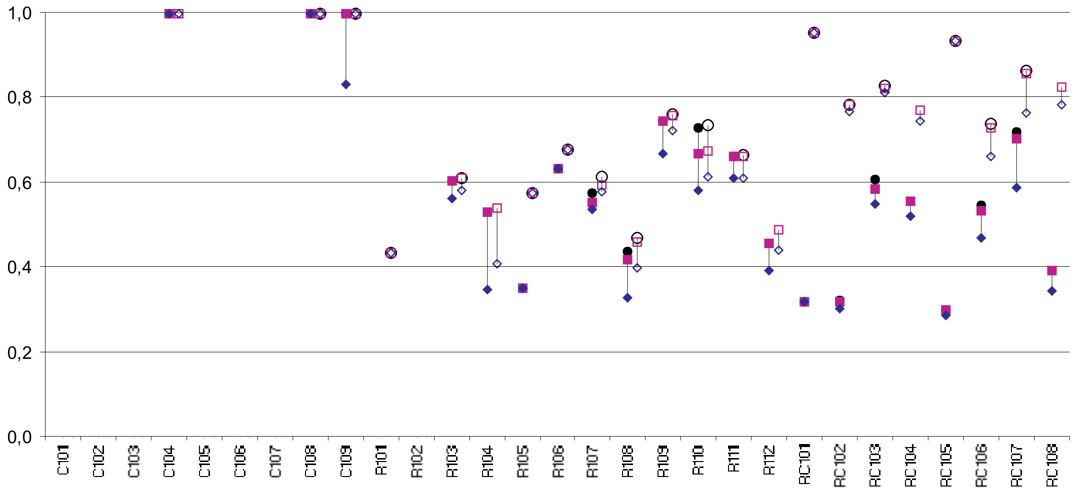

Figure 4: Results for the Solomon Problems with 100 Customers and *Short* Routes

1.3.2. Extensive Computational Test

The lower bounds and upper bounds given in the Tables 1–3 are useful to select some of the instances for applying the same algorithm without a time limit. Techniques for eliminating useless arcs from the graph underlying the SPPRC as described in Irnich (2004) allow for a speedup of the computations within the branch-and-bound search tree. Based on these acceleration techniques, Table 4 shows the information about optimal solutions for more than 15 previously unsolved instances from Solomon's benchmark problems. For instances marked with * different results have been reported in Cordeau et al. (2002). The problem c204.100 has been solved with 2-cycle elimination mainly because of extensive partial pricing

These results include four new optimal solutions for problems with short routes (series R1 and RC1). For these instances (r104.100, rc104.100, rc107.100, and rc108.100) 3-cycle elimination performs better than cycle-elimination for $k \geq 4$. The reason for this is that the fractional RMP solutions for $k = 3$ contain only a small portion of routes with cycles.

For the problems of series 2 higher values of k are sometime necessary to compute strong lower bounds. The instances rc202.100, rc203.50, and rc205.100 could only be solved with very long computation times using 5-cycle elimination.

Table 4: Optimal Solutions for Previously Unsolved VRPTW Instances.

Instance	Distance	#vehicles	k-cycle	Tree	Time (in s)
r104.100	971.5	11	3	5396	268106.0
rc104.100	1132.3	10	3	6757	986809.0
rc107.100	1207.8	12	3	1493	42770.7
rc108.100	1114.2	11	3	707	71263.0
c204.100	588.1	3	2	12	54254.4
r203.50	605.3	5	3	23	470.4
r204.25	355.0	2	4	35	231.7
r204.50	506.4	2	4	132	23749.5
r205.50*	690.1	4	4	137	1091.4
r206.50	632.4	4	3	1615	22455.3
r208.25*	328.2	1	3	16	363.5
r209.50	600.6	4	4	7	255.4
r210.50	645.6	4	3	960	11551.4
r211.50	535.5	3	3	1972	21323.0
rc202.50	613.6	5	4	28	503.3
rc202.100	1092.3	8	5	239	124018.0
rc203.25*	326.9	2	4	297	3455.3
rc203.50	555.3	4	5	38	54229.2
rc205.50*	630.2	5	4	5	82.4
rc205.100	1154.0	7	5	65	13295.9
rc206.50	610.0	5	4	62	934.9

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